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Combinatorial Bitstring Semantics for Arbitrary Logical Fragments

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Abstract Logical geometry systematically studies Aristotelian diagrams, such as the classical square of oppositions and its extensions. These investigations rely heavily on the use of bitstrings, which are compact combinatorial representations of formulas that allow us to quickly determine their Aristotelian relations. However, because of their general nature, bitstrings can be applied to a wide variety of topics in philosophical logic beyond those of logical geometry. Hence, the main aim of this paper is to present a systematic technique for assigning bitstrings to arbitrary finite fragments of formulas in arbitrary logical systems, and to study the logical and combinatorial properties of this technique. It is based on the partition of logical space that is induced by a given fragment, and sheds new light on a number of interesting issues, such as the logic-dependence of the Aristotelian relations and the subtle interplay between the Aristotelian and Boolean structure of logical fragments. Finally, the bitstring technique also allows us to systematically analyze fragments from contemporary logical systems, such as public announcement logic, which could not be done before.

Keywords bitstrings · combinatorial semantics · Aristotelian diagram · Boolean algebra · existential import · public announcement logic

1 Introduction

Throughout the history of philosophical logic, logicans have made use of Aristotelian diagrams to investigate and explain various subtle philosophical issues (Keynes, 1884;

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Khomskii, 2012; Parsons, 2006; Peckhaus, 2012; Read, 2012). These diagrams are also used in contemporary research in logic (Carnielli and Pizzi, 2008; Demey, 2012; Lenzen, 2012; McNamara, 2010; Mélès, 2012), and even in related fields such as computer science, cognitive science and linguistics (Ciucci et al, 2016; Horn, 1989; Mikhail, 2007; Seuren and Jaspers, 2014; van der Auwera, 1996; Yao, 2013). Nowadays, Aristotelian diagrams thus serve "as a kind of *lingua franca*" (Jacquette, 2012, p. 81) for an interdisciplinary research community on logical reasoning.

Regardless of this role as a lingua franca, the research project of logical geometry systematically studies Aristotelian diagrams as objects of independent interest, for example, in terms of their information content (Smessaert and Demey, 2014b).[1] Logical geometry makes extensive use of *bitstrings*: these are combinatorial representations of formulas that allow us to easily determine the Aristotelian relations holding between them (Smessaert, 2009; Demey and Smessaert, 2014; Smessaert and Demey, 2015b).

The main aim of this paper is to present, for the first time, a systematic technique for assigning bitstrings to any given finite fragment $\mathscr{F}$ of formulas in any logical system S, and to study its logical and combinatorial properties. This bitstring technique will be illustrated by means of a number of case studies coming from logical geometry itself; however, it should be emphasized that because of its general nature, this technique can be applied to a wide variety of topics in philosophical logic beyond those of logical geometry.

The paper is organized as follows. Section 2 introduces some logical preliminaries, and discusses the uses and limitations of the original, informal bitstring approach in logical geometry. Section 3 describes a systematic technique for assigning bitstrings to any given fragment, based on the partition induced by that fragment, and presents some numerical results on the correlation between fragment size and bitstring length. In order to illustrate the theoretical fruitfulness of this technique, Section 4 applies it to the logic-dependence of Aristotelian diagrams, whereas Section 5 applies it to the interaction between the Aristotelian and Boolean structure of logical fragments. Next, Section 6 uses the new bitstring technique to systematically analyze public announcement logic, which could not be done using the original bitstring approach. Finally, Section 7 wraps things up, and mentions some questions for further research.

2 The Bitstring Approach in Logical Geometry

2.1 Logical Preliminaries

Bitstrings provide a compact way of representing the semantics of the formulas in a given logical fragment or lexical field, and allow us to study the logical relations holding between these formulas in terms of their bitstring representations.[2] Although

[1] See `www.logicalgeometry.org`.

[2] A similar technique, that puts less emphasis on the semantics of the formulas, is Pellissier's (2008) *setting* approach. Seuren's (2010; 2013; 2014) *valuation spaces* and Schang's (2012a) *question-answer*

an informal precursor of this technique was already used by Avicenna in the 11th century AD (Chatti, 2012, 2014), its formal development began only in the last decade, inspired by considerations from generalized quantifier theory about partitioning the powerset of the quantificational domain (Smessaert, 2009). It has since been fruitfully applied to logical systems such as propositional logic, first-order logic and modal logic (Luzeaux et al, 2008; Smessaert, 2009; Smessaert and Demey, 2015c), and to lexical fields such as color terms, singular expressions and subjective quantification (Jaspers, 2012; Smessaert, 2012; Smessaert and Demey, 2015b).

From a mathematical perspective, a bitstring semantics of a finite fragment of formulas $\mathscr{F}$ consists of a Boolean algebra isomorphism $\beta\colon \mathbb{B}_n \to \{0,1\}^n$, where $\mathbb{B}_n$ is a Boolean algebra that contains $\mathscr{F}$. This isomorphism is always guaranteed to exist, by the representation theorem for finite Boolean algebras (Givant and Halmos, 2009, Corollary 15.1).[3] If $b \in \{0,1\}^n$ is a bitstring of length n, we will write $[b]_i$ to denote the bit in the i^{th} position of b (for $1 \leq i \leq n$); for example, if $b = 100 \in \{0,1\}^3$, we have $[b]_1 = 1$, $[b]_2 = 0$ and $[b]_3 = 0$. Bitstrings have mainly been used to study the Aristotelian relations holding between formulas, which are defined as follows.

Definition 1 (Aristotelian relations for a logical system) Let S be a logical system, which is assumed to have Boolean operators and a model-theoretic semantics $\models_{\mathsf{S}}$. The formulas φ and ψ are said to be

S-*contradictory*	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\models_{\mathsf{S}} \varphi \vee \psi$,
S-*contrary*	iff	$\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\not\models_{\mathsf{S}} \varphi \vee \psi$,
S-*subcontrary*	iff	$\not\models_{\mathsf{S}} \neg(\varphi \wedge \psi)$	and	$\models_{\mathsf{S}} \varphi \vee \psi$,
in S-*subalternation*	iff	$\models_{\mathsf{S}} \varphi \to \psi$	and	$\not\models_{\mathsf{S}} \psi \to \varphi$.

These relations are abbreviated as CD_{S}, C_{S}, SC_{S} and SA_{S}, respectively. The set $\mathscr{AG}_{\mathsf{S}}$ consisting of these four relations is called the *Aristotelian geometry of* S, i.e. $\mathscr{AG}_{\mathsf{S}} := \{CD_{\mathsf{S}}, C_{\mathsf{S}}, SC_{\mathsf{S}}, SA_{\mathsf{S}}\}$.[4]

Informally, the relations CD, C and SC are defined in terms of whether the formulas can be true together and whether they can be false together,[5] whereas SA is defined in terms of truth propagation (Smessaert and Demey, 2014b). Given the explicit Boolean structure involved in Definition 1, the Aristotelian relations can straightfor-

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semantics are also related, but mathematically less developed than the bitstring approach proposed in the present paper.

[3] Given Boolean algebras $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, \neg_{\mathbb{A}}, \bot_{\mathbb{A}}, \top_{\mathbb{A}}\rangle$ and $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \bot_{\mathbb{B}}, \top_{\mathbb{B}}\rangle$, a Boolean algebra isomorphism $f\colon \mathbb{A} \to \mathbb{B}$ is a bijection $f\colon A \to B$ that preserves Boolean structure, i.e. such that $f(x \wedge_{\mathbb{A}} y) = f(x) \wedge_{\mathbb{B}} f(y)$ and $f(\neg_{\mathbb{A}} x) = \neg_{\mathbb{B}} f(x)$. Using these properties, one can also show that $f(\bot_{\mathbb{A}}) = \bot_{\mathbb{B}}$, $f(\top_{\mathbb{A}}) = \top_{\mathbb{B}}$, $f(x \vee_{\mathbb{A}} y) = f(x) \vee_{\mathbb{B}} f(y)$, etc. Usually, we will omit the subscripts, and simply write $f(x \wedge y) = f(x) \wedge f(y)$, etc.

[4] When the system S is clear from the context, we will often leave it implicit, and simply talk about 'contrary' instead of 'S-contrary', and write C instead of C_{S}, etc.

[5] The $\neg(\varphi \wedge \psi)$ part in Definition 1 specifies whether the formulas can be true together; similarly, given the equivalence of $\varphi \vee \psi$ and $\neg(\neg\varphi \wedge \neg\psi)$, the $\varphi \vee \psi$ part in Definition 1 specifies whether the formulas can be false together.

wardly be generalized to arbitrary Boolean algebras, and in particular to bitstring algebras.[6]

Definition 2 (Aristotelian relations for bitstring algebras) Consider the Boolean algebra $\{0,1\}^n$, which consists of bitstrings of length n. The bitstrings b_1 and b_2 are said to be

n-contradictory	iff	$b_1 \wedge b_2 = \mathbf{0}_n$	and	$b_1 \vee b_2 = \mathbf{1}_n$,
n-contrary	iff	$b_1 \wedge b_2 = \mathbf{0}_n$	and	$b_1 \vee b_2 \neq \mathbf{1}_n$,
n-subcontrary	iff	$b_1 \wedge b_2 \neq \mathbf{0}_n$	and	$b_1 \vee b_2 = \mathbf{1}_n$,
in n-subalternation	iff	$b_1 \wedge b_2 = b_1$	and	$b_1 \vee b_2 \neq b_1$.

Again, these relations are abbreviated as CD_n, C_n, SC_n and SA_n, respectively, and the *Aristotelian geometry of bitstrings of length n* is $\mathscr{AG}_n := \{CD_n, C_n, SC_n, SA_n\}$. Furthermore, $\mathbf{0}_n$ and $\mathbf{1}_n$ denote the bottom and top elements of the Boolean algebra $\{0,1\}^n$, i.e. the bitstrings $\underbrace{0\cdots0}_{n \text{ bits}}$ and $\underbrace{1\cdots1}_{n \text{ bits}}$, respectively.[7]

Since the Aristotelian relations are defined in purely Boolean terms, two sets of formulas/bitstrings that have the same Boolean structure will also have the same Aristotelian structure. Furthermore, since a bitstring semantics is itself a Boolean algebra isomorphism, the Aristotelian structure of a logical fragment is fully captured by its bitstring representation.[8]

Definition 3 (Boolean closure of a fragment) Let $\mathscr{F}$ be a finite set of formulas of some logical system S or bitstrings of some given length n. The *Boolean closure of* $\mathscr{F}$, denoted $\mathbb{B}(\mathscr{F})$, is the smallest Boolean algebra that contains $\mathscr{F}$, i.e. $\mathscr{F} \subseteq \mathbb{B}(\mathscr{F})$ and for all Boolean algebras $\mathbb{B}$ such that $\mathscr{F} \subseteq \mathbb{B}$, it holds that $\mathbb{B}(\mathscr{F}) \subseteq \mathbb{B}$.

Definition 4 (Aristotelian and Boolean isomorphism) Consider logical systems $\mathsf{S}_1, \mathsf{S}_2$ and natural numbers n_1, n_2. Consider logical systems and/or natural numbers $x \in \{\mathsf{S}_1, n_1\}$ and $y \in \{\mathsf{S}_2, n_2\}$. Let $\mathscr{F}_x$ be a set of formulas of system x or bitstrings of length x, and let $\mathscr{F}_y$ be a set of formulas of system y or bitstrings of length y. A bijection $\gamma\colon \mathscr{F}_x \to \mathscr{F}_y$ is said to be

- an *Aristotelian isomorphism* iff for all Aristotelian relations $R_x \in \mathscr{AG}_x$ and corresponding $R_y \in \mathscr{AG}_y$,[9] and for all $\varphi, \psi \in \mathscr{F}_x$, it holds that $R_x(\varphi,\psi)$ iff $R_y(\gamma(\varphi),\gamma(\psi))$,

[6] It should be emphasized that unlike Moretti (2012) and Schang (2012b), who use bitstrings to encode the Aristotelian *relations* holding between formulas, the current paper uses bitstrings to encode the *formulas* themselves.

[7] Again, when the bitstring length n is clear from the context, we will often leave it implicit, and simply talk about 'contrary' instead of 'n-contrary', and write C instead of C_n, etc.

[8] To avoid terminological confusion, note that we use the term *Boolean isomorphism* for a mapping between two *fragments*, regardless of their size and/or Boolean structure (see Definition 4). By contrast, the term *Boolean algebra isomorphism* is used for a mapping between two *Boolean algebras* (this is the usual notion of a bijective homomorphism between Boolean algebras; see, for example, Givant and Halmos (2009, Chapter 12)).

[9] For example, if R_x is x-contrariety, then R_y is the corresponding relation of y-contrariety.

– a *Boolean isomorphism* iff there exists some Boolean algebra isomorphism $f\colon \mathbb{B}(\mathscr{F}_x) \to \mathbb{B}(\mathscr{F}_y)$ such that $\gamma = f \upharpoonright \mathscr{F}_x$.[10]

Lemma 1 *Let $\mathscr{F}_x, \mathscr{F}_y$ be as in Definition 4, and consider a function $\gamma\colon \mathscr{F}_x \to \mathscr{F}_y$. If γ is a Boolean isomorphism, then γ is an Aristotelian isomorphism.*

Proof For the sake of concreteness, we will assume that x is some logical system S, and y is some natural number n; the other three cases (see Footnote 10) are completely analogous. Furthermore, we will only deal with the case of contrariety; the three other Aristotelian relations are also completely analogous. We thus need to show that $C_\mathsf{S}(\varphi, \psi)$ iff $C_n(\gamma(\varphi), \gamma(\psi))$, for all $\varphi, \psi \in \mathscr{F}_x$.

Since γ is a Boolean isomorphism, there exists a Boolean algebra isomorphism $f\colon \mathbb{B}(\mathscr{F}_x) \to \mathbb{B}(\mathscr{F}_y)$ such that $\gamma = f \upharpoonright \mathscr{F}_x$. Hence, for all $\varphi, \psi \in \mathscr{F}_x$, we have the following chain of equivalences:

$$\begin{array}{llccc}
C_\mathsf{S}(\varphi,\psi) & \Leftrightarrow & \models_\mathsf{S} \neg(\varphi\wedge\psi) & \text{and} & \not\models_\mathsf{S} \varphi\vee\psi \\
 & \Leftrightarrow & \varphi\wedge\psi \equiv_\mathsf{S} \bot & \text{and} & \varphi\vee\psi \not\equiv_\mathsf{S} \top \\
 & \Leftrightarrow & f(\varphi\wedge\psi) = f(\bot) & \text{and} & f(\varphi\vee\psi) \neq f(\top) \\
 & \overset{*}{\Leftrightarrow} & f(\varphi)\wedge f(\psi) = \mathbf{0}_n & \text{and} & f(\varphi)\vee f(\psi) \neq \mathbf{1}_n \\
 & \overset{\circ}{\Leftrightarrow} & \gamma(\varphi)\wedge\gamma(\psi) = \mathbf{0}_n & \text{and} & \gamma(\varphi)\vee\gamma(\psi) \neq \mathbf{1}_n \\
 & \Leftrightarrow & & C_n(\gamma(\varphi),\gamma(\psi)). &
\end{array}$$

The first and last steps merely apply Definitions 1 and 2, respectively. The crucial $*$-labeled step is justified by the fact that f is a Boolean algebra isomorphism. The $\circ$-labeled step, finally, holds because $\gamma = f \upharpoonright \mathscr{F}_x$ and $\varphi, \psi \in \mathscr{F}_x$; we thus prove something about γ by taking a 'detour' via the Boolean algebra isomorphism f. □

Lemma 2 *Consider a fragment $\mathscr{F}$ of formulas of some logical system S, and let $\mathbb{B}$ be a finite Boolean algebra (containing 2^n formulas) such that $\mathscr{F} \subseteq \mathbb{B}$. Every bitstring semantics $\beta\colon \mathbb{B} \to \{0,1\}^n$ is an Aristotelian isomorphism.*

Proof Completely analogous to the proof of Lemma 1; however, since β is itself already a Boolean algebra isomorphism, we can use it directly, rather than having to take a 'detour' via another function. □

Lemma 2 allows us to systematically study Aristotelian diagrams for given logical systems *via* their bitstring representations. Consider, for example, the fragment $\mathscr{F} = \{\Box p, \Diamond p, \Box\neg p, \Diamond\neg p\}$ from the modal logic S5. A bitstring semantics β_1 for this fragment is generated by $\beta_1(\Box p) = 100$, $\beta_1(\Diamond p) = 110$, $\beta_1(\Box\neg p) = 001$ and $\beta_1(\Diamond\neg p) = 011$. An alternative bitstring semantics β_2 makes use of bitstrings of length 4, and is generated by $\beta_2(\Box p) = 1000$, $\beta_2(\Diamond p) = 1110$, $\beta_2(\Box\neg p) = 0001$ and $\beta_2(\Diamond\neg p) = 0111$. Using β_1 and β_2, the Aristotelian square of oppositions for $\mathscr{F}$ in Figure 1(a) can now straightforwardly be mapped onto the bitstring squares in Figure 1(b) and 1(c), respectively (Smessaert, 2009).

[10] Depending on the values of x and y, there are thus four 'types' of Aristotelian and Boolean isomorphisms: (i) from formulas of S_1 to formulas of S_2, (ii) from formulas of S_1 to bitstrings of length n_2, (iii) from bitstrings of length n_1 to formulas of S_2, and (iv) from bitstrings of length n_1 to bitstrings of length n_2. Finally, note that it immediately follows from this definition that if two fragments are Boolean isomorphic, they have the same Boolean closure (up to Boolean algebra isomorphism).

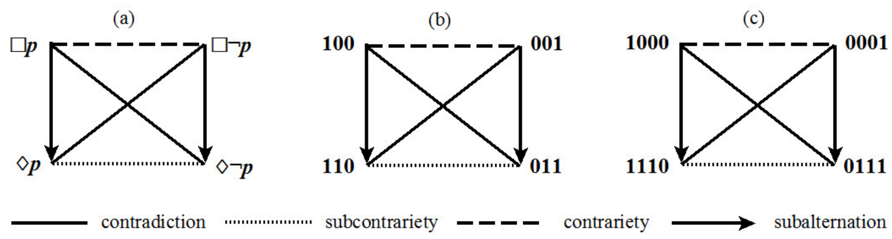

Fig. 1 (a) Aristotelian square for S5, (b) bitstring square with bitstrings of length 3, (c) bitstring square with bitstrings of length 4.

2.2 Applications of the Bitstring Approach

In recent years, the bitstring approach has proved to be very fruitful, and yielded a wide variety of logical and diagrammatic results. For example, in previous work we have studied the subtle interaction between bitstring properties and logical notions such as unconnectedness. Two formulas are said to be *unconnected* iff they do not stand in any Aristotelian relation.[11] It can be shown[12] that unconnectedness requires bitstrings of length at least 4: if $b_1, b_2 \in \{0,1\}^n$ are unconnected, then $n \geq 4$. Hence, if two formulas are unconnected, they have to be represented by bitstrings of length at least 4. Consequently, if an Aristotelian diagram can be encoded by bitstrings of length 3, then it cannot contain any unconnectedness, i.e. every pair of its formulas stands in some Aristotelian relation. For example, the strong Jacoby-Sesmat-Blanché (JSB) hexagon for S5 in Figure 2(a) can be encoded by bitstrings of length 3, and thus does not contain any unconnectedness. By contrast, the unconnected-4 (U4) hexagon in Figure 2(c) can only be encoded by bitstrings of length at least 4, and does contain unconnectedness (e.g. the formulas p and $\Diamond p \wedge \Diamond\neg p$ are unconnected). Finally, it should be noted that there are also Aristotelian diagrams that can only be encoded by bitstrings of length at least 4, and yet do not contain any unconnectedness; for example, see the Sherwood-Czeżowski (SC) hexagon in Figure 2(b). The overall situation is summarized in the following table:[13]

	no unconnectedness	unconnectedness
length 3 required	strong Jacoby-Sesmat-Blanché	—
length 4 required	Sherwood-Czeżowski	unconnected-4

[11] Many authors refer to this same notion as *logical independence*, e.g. see Hughes (1987), Béziau (2003), Seuren (2010) and Read (2012). Furthermore, Smessaert and Demey (2014b) provide an alternative, positive characterization of unconnectedness as the combination of two other relations, viz. non-contradiction and non-implication.

[12] See Smessaert and Demey (2014b, 2017) for the original formulation and proof of this theorem.

[13] The JSB hexagon in Figure 2(a) is named after Jacoby (1950), Sesmat (1951) and Blanché (1966), and the SC hexagon in (b) after William of Sherwood (Kretzmann, 1966; Khomskii, 2012) and Czezowski (1955). The Boolean differences between these two types of diagrams are studied in Smessaert (2012). The distinction between *strong* and *weak* JSB hexagons was introduced by Pellissier (2008), and will be studied in more detail in Subsection 5.1. Finally, the diagram in Figure 2(c) is called an 'unconnected-4' hexagon because it contains exactly 4 pairs of unconnected formulas; it has recently been studied by Seuren (2013) and Smessaert and Demey (2014a).

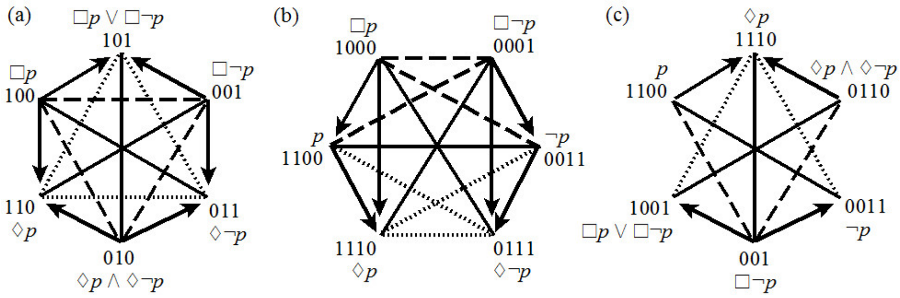

Fig. 2 Three Aristotelian hexagons for S5: (a) strong Jacoby-Sesmat-Blanché (JSB), (b) Sherwood-Czeżowski (SC), (c) unconnected-4 (U4).

Furthermore, bitstrings have also proved extremely useful in studying the interactions between various Aristotelian diagrams. The JSB hexagon in Figure 2(a) is *Boolean closed*: every contingent Boolean combination of formulas in this hexagon is (logically equivalent to) a formula that already belongs to it.[14] It thus visualizes the 'entire' Boolean algebra $\mathbb{B}_3$ (represented by bitstrings of length 3), except for its $\top$- and $\bot$-elements. In exactly the same way, $\mathbb{B}_4$ (represented by bitstrings of length 4) can be visualized by means of a three-dimensional Aristotelian diagram, viz. the *rhombic dodecahedron* (RDH). The latter has been shown to contain many well-known Aristotelian diagrams, such as 10 JSB hexagons (6 strong and 4 weak ones), 12 SC hexagons, 12 U4 hexagons, 6 so-called Buridan octagons, and many others.[15] Furthermore, there exist complementarities between these types of subdiagrams; for example, if we consider a strong JSB hexagon inside RDH, the formulas that do not belong to this hexagon turn out to constitute an interesting Aristotelian subdiagram by themselves, viz. a Buridan octagon (Smessaert, 2009; Demey and Smessaert, 2014; Smessaert and Demey, 2014a, 2015b,c).

These geometric results can all be captured in terms of bitstrings. If we consider two bit positions, for example the second and third, then the 14 contingent bitstrings of length 4 can be partitioned into a group of 6 having identical values in those positions, and a group of 8 having different values in those positions.[16] These two groups constitute a strong JSB hexagon and its complementary Buridan octagon, respectively. This should not be surprising: although we are dealing with bitstrings of length

[14] Since Aristotelian diagrams typically contain only contingent formulas, the definition of being Boolean closed is restricted to *contingent* Boolean combinations. For example, even if a diagram contains formulas φ and $\neg\varphi$, the condition of being Boolean closed does *not* require that it also contain their tautological disjunction $\varphi \vee \neg\varphi$ and contradictory conjunction $\varphi \wedge \neg\varphi$.

[15] A Buridan octagon is a type of Aristotelian diagram that was first studied by the medieval philosopher John Buridan (Hughes, 1987; Read, 2012), and can be shown to contain two SC and two U4 hexagons (Smessaert and Demey, 2014a, 2015c). We will return to this type of Aristotelian diagram in Subsection 5.2.

[16] The 6 bitstrings with identical values are 1**00**1, 1**00**0, 0**00**1, 1**11**0, 0**11**1 and 0**11**0; the 8 bitstrings with different values are 1**10**1, 1**10**0, 0**10**1, 0**10**0, 1**01**1, 1**01**0, 0**01**1 and 0**01**0. Of course, the top- and bottom elements 1**11**1 and 0**00**0 also have identical values in their second and third bit positions, but as usual, these are ignored in Aristotelian diagrams (recall Footnote 14), which explains the numerical discrepancy between the two groups.

4, the bitstrings in the first group have identical values in two of their bit positions, and can thus be 'compressed' into bitstrings of length 3, which constitute the JSB hexagon in Figure 2(a).[17] There are exactly 6 ways in which bitstrings of length 4 can have identical (resp. different) values in two of their bit positions, and these correspond exactly to the 6 strong JSB hexagons (resp. Buridan octagons) inside RDH (Smessaert and Demey, 2017).[18]

Bitstrings have also been used for a variety of other purposes. For example, they have led to some combinatorial results on the numbers of Aristotelian relations in arbitrary Boolean algebras, which serve as a motivation for the account of diagram informativity presented in Smessaert and Demey (2014b). A more exhaustive overview of the applications of bitstrings is presented in Smessaert and Demey (2017).

2.3 Limitations of the Bitstring Approach

We have just seen that bitstrings have proved to be extremely useful in logical geometry. Still, it cannot be denied that bitstring mappings exhibit a certain degree of arbitrariness. For example, the concrete order of the bit positions does not seem to matter much: a formula such as $\Box p$ is usually mapped onto the bitstring 100 because of cognitive and linguistic reasons (Smessaert, 2009; Seuren and Jaspers, 2014), but from a purely mathematical perspective, nothing prevents us from swapping the first two bit positions and representing this same formula by means of 010.[19] However, there is also a certain kind of arbitrariness to the bitstring approach which is of a more fundamental nature, and therefore seriously restricts its usefulness. We will now discuss three such limitations in more detail.

First of all, it is not always clear how 'sensitive' bitstrings are to the specific properties of the underlying logical system. For example, given two formulas φ, ψ and two logical systems S_1 and S_2, it is perfectly conceivable that φ and ψ are S_1-*contradictory*, but S_2-*contrary*. However, if we assign bitstrings $\beta(\varphi)$ and $\beta(\psi)$ to these formulas, at most one of these Aristotelian relations can be captured: (i) if $\beta(\varphi)$ and $\beta(\psi)$ are contradictory, the S_1-contradiction is captured but the S_2-contrariety is not, (ii) if $\beta(\varphi)$ and $\beta(\psi)$ are contrary, the S_2-contrariety is captured but the S_1-contradiction is not, and (iii) if $\beta(\varphi)$ and $\beta(\psi)$ are subcontrary, in subalternation or unconnected, then neither the S_1-contradiction nor the S_2-contrariety is captured. Note, in particular, that by Definition 2 it cannot be the case that the bitstrings

[17] For example, by collapsing the second and third bit positions, the bitstrings 1**00**0 and 0**11**0 for $\Box p$ and $\Diamond p \wedge \Diamond \neg p$ in RDH are compressed into the bitstrings 1**0**0 and 0**1**0 in Figure 2(a), respectively.

[18] We will write $[b]_i = [b]_j$ to express the condition that a bitstring b has the *same* value in bit positions i and j. The complementary condition $[b]_i \neq [b]_j$ is satisfied by bitstrings with *different* values in positions i and j. Using this notation, the 6 strong JSB hexagons inside RDH correspond to the conditions $[b]_1 = [b]_2$, $[b]_1 = [b]_3$, $[b]_1 = [b]_4$, $[b]_2 = [b]_3$, $[b]_2 = [b]_4$ and $[b]_3 = [b]_4$, and the 6 complementary Buridan octagons correspond to the complementary conditions $[b]_1 \neq [b]_2$, $[b]_1 \neq [b]_3$, $[b]_1 \neq [b]_4$, $[b]_2 \neq [b]_3$, $[b]_2 \neq [b]_4$ and $[b]_3 \neq [b]_4$. Other subdiagrams of RDH turn out to correspond to other, more complex conditions on bitstrings.

[19] In general, it holds for all bitstrings $b_1, b_2 \in \{0,1\}^n$, Aristotelian relations R and permutations $\pi\colon \{0,1\}^n \to \{0,1\}^n$ that $R(b_1,b_2)$ iff $R(\pi(b_1),\pi(b_2))$. Hence, if a logical fragment $\mathscr{F}$ can be represented by means of a bitstring isomorphism $\beta : \mathscr{F} \subseteq \mathbb{B}_n \to \{0,1\}^n$, it can equally well be represented by means of $\pi \circ \beta : \mathscr{F} \subseteq \mathbb{B}_n \to \{0,1\}^n$.

$\beta(\varphi)$ and $\beta(\psi)$ are simultaneously contradictory and contrary. Intuitively, the solution consists in using two different bitstring isomorphisms β_1 and β_2, which allows for the possibility that $\beta_1(\varphi)$ and $\beta_1(\psi)$ are contradictory (which captures the S_1-contradiction of φ and ψ), while $\beta_2(\varphi)$ and $\beta_2(\psi)$ are contrary (which captures the S_2-contrariety of φ and ψ). However, in the existing literature there have been no attempts to develop this intuitive idea in a systematic way.

A second problem concerns the nature of the interplay between Boolean and Aristotelian structure. We have already seen above that Aristotelian structure is determined by Boolean structure, and thus, that every Boolean isomorphism is also an Aristotelian isomorphism (Lemma 1). However, as will be illustrated later, it may be the case that a bijection $\gamma\colon \mathscr{F}_1 \to \mathscr{F}_2$ is an Aristotelian, but not a Boolean isomorphism. For example, it can happen that there are $\varphi_1, \varphi_2, \varphi_3 \in \mathscr{F}_1$ such that $\varphi_3 \equiv \varphi_1 \wedge \varphi_2$, and yet $\gamma(\varphi_3) \not\equiv \gamma(\varphi_1) \wedge \gamma(\varphi_2)$. Despite these fragments being Aristotelian isomorphic, there do not exist bitstring semantics $\beta_1\colon \mathscr{F}_1 \subseteq \mathbb{B}_n \to \{0,1\}^n$ and $\beta_2\colon \mathscr{F}_2 \subseteq \mathbb{B}_n \to \{0,1\}^n$ that map the formula φ_i and its counterpart $\gamma(\varphi_i)$ onto the same bitstring $b_i \in \{0,1\}^n$—so $\beta_1(\varphi_i) = b_i = \beta_2(\gamma(\varphi_i))$ for $1 \leq i \leq 3$. After all, either it holds that $b_3 = b_1 \wedge b_2$ (in which case β_2 is not a Boolean isomorphism), or $b_3 \neq b_1 \wedge b_2$ (in which case β_1 is not a Boolean isomorphism). The differences between fragments such as $\mathscr{F}_1$ and $\mathscr{F}_2$ in terms of their *Boolean closure* will turn out to play a crucial role in obtaining adequate bitstring semantics for them.

The third and final problem of the bitstring approach concerns its apparent lack of systematicity. Although it has been applied successfully to study the Aristotelian geometry of important logical systems and lexical fields (such as first-order logic, modal logic, subjective quantification, etc.), it cannot straightforwardly be generalized to new logical systems and/or fragments. For example, Demey (2012, 2014) studied in detail the Aristotelian geometry $\mathscr{AG}_{\mathsf{PAL}}$ of public announcement logic, but had to leave the task of obtaining a bitstring semantics for this system as an open problem. Ideally, however, the bitstring approach should provide a systematic strategy for establishing a bitstring semantics for any fragment $\mathscr{F}$ of any logical system S, and to study the Aristotelian geometry $\mathscr{AG}_{\mathsf{S}}$ in terms of its bitstring counterpart $\mathscr{AG}_n$.

In Sections 4, 5 and 6, we will return to these three limitations of the current bitstring approach, and show that they are all manifestations of a single underlying issue, viz. the need to compute the full Boolean closure of a given fragment before its bitstring semantics can be defined. This is particularly problematic from a practical perspective: we typically want to use a fragment's bitstring semantics as a compact representation of the essential properties of its Boolean closure, but this does not make much sense if we first have to calculate the full Boolean closure anyway. Therefore, in the next section, we will present a new systematic technique for obtaining a bitstring semantics for any logical fragment, which does *not* require calculating its full Boolean closure first.

3 Partition-Based Bitstring Semantics

3.1 Partitions Induced by Logical Fragments

As was noted in Section 2, the bitstring approach was originally inspired by considerations about partitioning domains of quantification (Smessaert, 2009). We will now show that this connection with partitions can be exploited to develop bitstring semantics into a more efficient and systematic technique.

Definition 5 (partition induced by a logical fragment) Let S be a logical system, which is assumed to have Boolean operators and a model-theoretic semantics $\models_{\mathsf{S}}$, and let $\mathscr{F} = \{\varphi_1, \ldots, \varphi_m\} \subseteq \mathscr{L}_{\mathsf{S}}$ be a finite fragment of the language of S. The *partition of* S *induced by* $\mathscr{F}$ is

$$\Pi_{\mathsf{S}}(\mathscr{F}) := \{\alpha \in \mathscr{L}_{\mathsf{S}} \mid \alpha \equiv_{\mathsf{S}} \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m, \text{ and } \alpha \text{ is } \mathsf{S}\text{-consistent}\}.$$

In this definition, $\pm\varphi$ stands for either φ or $\neg\varphi$. Furthermore, the formulas $\alpha \in \Pi_{\mathsf{S}}(\mathscr{F})$ will be called *anchor formulas*.

Each anchor formula is thus equivalent to a conjunction consisting of $m = |\mathscr{F}|$ conjuncts. In many circumstances (for example when $\neg\varphi_i \equiv_{\mathsf{S}} \varphi_j$ for some $\varphi_i, \varphi_j \in \mathscr{F}$), these conjunctions can be simplified.

Example 1 Consider the system of first-order logic (FOL) and the fragment $\mathscr{F}^\dagger := \{\forall xPx,\ \exists xPx,\ \neg Pa\}$. There are $2^{|\mathscr{F}^\dagger|} = 8$ relevant conjunctions, but 4 of them are FOL-inconsistent, and the 4 others can be simplified:

1.	$\forall xPx$	$\wedge$	$\exists xPx$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	inconsistent
2.	$\forall xPx$	$\wedge$	$\exists xPx$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	$\forall xPx$
3.	$\forall xPx$	$\wedge$	$\neg\exists xPx$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	inconsistent
4.	$\forall xPx$	$\wedge$	$\neg\exists xPx$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	inconsistent
5.	$\neg\forall xPx$	$\wedge$	$\exists xPx$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	$\neg Pa \wedge \exists xPx$
6.	$\neg\forall xPx$	$\wedge$	$\exists xPx$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	$Pa \wedge \neg\forall xPx$
7.	$\neg\forall xPx$	$\wedge$	$\neg\exists xPx$	$\wedge$	$\neg Pa$	$\rightsquigarrow$	$\neg\exists xPx$
8.	$\neg\forall xPx$	$\wedge$	$\neg\exists xPx$	$\wedge$	$\neg\neg Pa$	$\rightsquigarrow$	inconsistent

And hence, $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger) = \{\forall xPx,\ Pa \wedge \neg\forall xPx,\ \neg Pa \wedge \exists xPx,\ \neg\exists xPx\}$.

Example 2 Consider the system of classical propositional logic (CPL) and the fragment $\mathscr{F}^\ddagger := \{p \wedge q,\ \neg p \vee \neg q,\ p \vee q,\ \neg p \wedge \neg q\}$. There are $2^{|\mathscr{F}^\ddagger|} = 16$ relevant conjunctions; 13 are CPL-inconsistent, and the remaining 3 can be simplified,[20] thus yielding $\Pi_{\mathsf{CPL}}(\mathscr{F}^\ddagger) = \{p \wedge q, (p \vee q) \wedge (\neg p \vee \neg q),\ \neg p \wedge \neg q\}$.

The set $\Pi_{\mathsf{S}}(\mathscr{F})$ can be seen as a partition of the class of all models of the system S, and the anchor formulas as the cells of this partition. In particular, the cells are mutually exclusive and jointly exhaustive:

[20] For example, the conjunction $(p \wedge q) \wedge (\neg p \vee \neg q) \wedge (p \vee q) \wedge (\neg p \wedge \neg q)$ is CPL-inconsistent, while the conjunction $(p \wedge q) \wedge \neg(\neg p \vee \neg q) \wedge (p \vee q) \wedge \neg(\neg p \wedge \neg q)$ can be simplified to $p \wedge q$.

Lemma 3 *Given $\mathscr{F} = \{\varphi_1, \ldots, \varphi_m\}$ and $\Pi_{\mathsf{S}}(\mathscr{F}) = \{\alpha_1, \ldots, \alpha_n\}$, it holds that*

1. $\models_{\mathsf{S}} \neg(\alpha_i \wedge \alpha_j)$, *for* $1 \leq i \neq j \leq n$ *(mutually exclusive)*
2. $\models_{\mathsf{S}} \bigvee_{i=1}^{i=n} \alpha_i$ *(jointly exhaustive)*

Proof 1. Consider anchor formulas α_i, α_j, with $i \neq j$, i.e. $\alpha_i \not\equiv_{\mathsf{S}} \alpha_j$. By definition, these are equivalent to conjunctions c_i and c_j of the form $\pm\varphi_1 \wedge \cdots \pm \varphi_m$. There is at least one $\varphi_k \in \mathscr{F}$ such that φ_k occurs as a conjunct in c_i and $\neg\varphi_k$ occurs as a conjunct in c_j, or vice versa (after all, if there were no such φ_k, then $\alpha_i \equiv_{\mathsf{S}} c_i = c_j \equiv_{\mathsf{S}} \alpha_j$). It follows that $\alpha_i \wedge \alpha_j \equiv_{\mathsf{S}} c_i \wedge c_j = (\cdots \wedge \varphi_k \wedge \cdots) \wedge (\cdots \wedge \neg\varphi_k \wedge \cdots) \equiv_{\mathsf{S}} \bot$, and thus $\models_{\mathsf{S}} \neg(\alpha_i \wedge \alpha_j)$.

2. Consider an arbitrary S-model M. Define the formula $\alpha_M := (\varphi_1)_M \wedge \cdots \wedge (\varphi_m)_M$, where $(\varphi)_M := \varphi$ if $M \models \varphi$, and $(\varphi)_M := \neg\varphi$ if $M \not\models \varphi$. By definition, it holds that $M \models (\varphi_i)_M$ for all $1 \leq i \leq m$, and thus also $M \models \alpha_M$. Hence, α_M is consistent, and thus $\alpha_M \in \Pi_{\mathsf{S}}(\mathscr{F})$, i.e. $\alpha_M = \alpha_j$ for some $1 \leq j \leq n$. From $M \models \alpha_j$, it trivially follows that $M \models \bigvee_{i=1}^{i=n} \alpha_i$. □

In Definition 5, the partition $\Pi_{\mathsf{S}}(\mathscr{F})$ is defined immediately for the entire fragment $\mathscr{F}$. However, in many concrete cases, it can be useful to construct this partition in a more 'incremental' fashion. This stepwise approach involves first 'decomposing' the fragment $\mathscr{F}$ into two (or more) subfragments $\mathscr{F}_1$ and $\mathscr{F}_2$,[21] and then defining the partitions $\Pi_{\mathsf{S}}(\mathscr{F}_1)$ and $\Pi_{\mathsf{S}}(\mathscr{F}_2)$ that are induced by these subfragments. Finally, the meet of these two partitions can be shown to coincide with the original partition induced by the full fragment.[22]

Definition 6 (meet of partitions) Given partitions Π_1 and Π_2, we define:

$$\Pi_1 \wedge_{\mathsf{S}} \Pi_2 := \{\gamma_1 \wedge \gamma_2 \mid \gamma_1 \in \Pi_1,\ \gamma_2 \in \Pi_2, \text{ and } \gamma_1 \wedge \gamma_2 \text{ is } \mathsf{S}\text{-consistent}\}.$$

Lemma 4 *If $\mathscr{F}_1 \cup \mathscr{F}_2 = \mathscr{F}$, then $\Pi_{\mathsf{S}}(\mathscr{F}_1) \wedge_{\mathsf{S}} \Pi_{\mathsf{S}}(\mathscr{F}_2) = \Pi_{\mathsf{S}}(\mathscr{F})$.*

Proof Without loss of generality, we assume that $\mathscr{F} = \{\varphi_1, \ldots, \varphi_k, \varphi_{k+1}, \ldots, \varphi_m\}$ is decomposed into $\mathscr{F}_1 = \{\varphi_1, \ldots, \varphi_k\}$ and $\mathscr{F}_2 = \{\varphi_{k+1}, \ldots, \varphi_m\}$. For all formulas γ, it holds that $\gamma \in \Pi_{\mathsf{S}}(\mathscr{F}_1) \wedge_{\mathsf{S}} \Pi_{\mathsf{S}}(\mathscr{F}_2)$

$$\begin{aligned}
&\Leftrightarrow \quad \exists \gamma_1 \in \Pi_{\mathsf{S}}(\mathscr{F}_1), \gamma_2 \in \Pi_{\mathsf{S}}(\mathscr{F}_2)\colon\ \gamma \equiv_{\mathsf{S}} \gamma_1 \wedge \gamma_2,\ \gamma \text{ is } \mathsf{S}\text{-consistent} \\
&\Leftrightarrow \quad \gamma \equiv_{\mathsf{S}} (\pm\varphi_1 \wedge \cdots \wedge \pm\varphi_k) \wedge (\pm\varphi_{k+1} \wedge \cdots \wedge \pm\varphi_n),\ \gamma \text{ is } \mathsf{S}\text{-consistent} \\
&\Leftrightarrow \quad \gamma \in \Pi_{\mathsf{S}}(\mathscr{F}).
\end{aligned}$$

The first equivalence is justified by Definition 6; the second and third equivalences are justified by Definition 5. □

[21] We thus require that $\mathscr{F}_1 \cup \mathscr{F}_2 = \mathscr{F}$. In concrete cases it will also typically hold that $\mathscr{F}_1 \cap \mathscr{F}_2 = \emptyset$, but this is not strictly necessary.

[22] From a mathematical perspective, the meet of two partitions is well-defined because partitions form a lattice structure (Grätzer, 1978; Canfield, 2001). From a more cognitive perspective, meets of partitions also play an important role in concept formation; for example, Seuren and Jaspers (2014, p. 627) describe how the partition {male, female} "crosscuts" the partition {minor, adult}, thereby producing the new, more fine-grained partition {boy, man, girl, woman}. Finally, see Carroll (1977) and Moretti (2014) for a more visual-diagrammatic perspective on crosscutting partitions.

Lemma 5 *If $\mathscr{F}_1 \subseteq \mathscr{F}_2$, then $\Pi_{\mathsf{S}}(\mathscr{F}_2)$ is a refinement of $\Pi_{\mathsf{S}}(\mathscr{F}_1)$; i.e. for every $\alpha_2 \in \Pi_{\mathsf{S}}(\mathscr{F}_2)$ there exists an $\alpha_1 \in \Pi_{\mathsf{S}}(\mathscr{F}_1)$ such that $\models_{\mathsf{S}} \alpha_2 \to \alpha_1$.*

Proof Applying Lemma 4, we see that $\Pi_{\mathsf{S}}(\mathscr{F}_2) = \Pi_{\mathsf{S}}(\mathscr{F}_1 \cup (\mathscr{F}_2 - \mathscr{F}_1)) = \Pi_{\mathsf{S}}(\mathscr{F}_1) \wedge_{\mathsf{S}} \Pi_{\mathsf{S}}(\mathscr{F}_2 - \mathscr{F}_1)$, and thus every anchor formula $\alpha_2 \in \Pi_{\mathsf{S}}(\mathscr{F}_2)$ is of the form $\alpha_1 \wedge \gamma$, for some $\alpha_1 \in \Pi_{\mathsf{S}}(\mathscr{F}_1)$ and $\gamma \in \Pi_{\mathsf{S}}(\mathscr{F}_2 - \mathscr{F}_1)$. Trivially, then, $\models_{\mathsf{S}} \alpha_2 \to \alpha_1$. □

Before moving from partitions to bitstrings, we will prove two final results about partitions that will turn out to be useful later on. It should be noted that these results are not simply about the fragment $\mathscr{F}$, but rather about its entire Boolean closure $\mathbb{B}(\mathscr{F})$.

Lemma 6 *Given a fragment $\mathscr{F} = \{\varphi_1, \ldots, \varphi_m\} \subseteq \mathscr{L}$, the following holds:*

for all $\varphi \in \mathbb{B}(\mathscr{F})$, for all $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$: $\models_{\mathsf{S}} \alpha_i \to \varphi$ or $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$, but not both.

Proof The 'not both'-part is trivial, since every $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$ is, by Definition 5, S-consistent. The 'or'-part is proved by induction on the complexity of φ.

Base case: $\varphi \in \mathscr{F}$. Hence, $\varphi = \varphi_j$ for some $1 \leq j \leq m$. For every $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$, it holds by Definition 5 that $\alpha_i \equiv_{\mathsf{S}} \pm\varphi_1 \wedge \cdots \wedge \pm\varphi_j \wedge \cdots \wedge \pm\varphi_m$. If $\pm\varphi_j = \varphi_j$, then $\models_{\mathsf{S}} \alpha_i \to \varphi$; if $\pm\varphi_j = \neg\varphi_j$, then $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$.

Induction case for negation: $\varphi = \neg\psi$. The induction hypothesis is that for all $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$: $\models_{\mathsf{S}} \alpha_i \to \psi$ or $\models_{\mathsf{S}} \alpha_i \to \neg\psi$. In the former case, it follows that $\models_{\mathsf{S}} \alpha_i \to \neg\neg\psi$, i.e. $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$; the latter case means that $\models_{\mathsf{S}} \alpha_i \to \varphi$.

Induction case for conjunction: $\varphi = \psi_1 \wedge \psi_2$. The induction hypotheses are that for all $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$: ($\models_{\mathsf{S}} \alpha_i \to \psi_1$ or $\models_{\mathsf{S}} \alpha_i \to \neg\psi_1$) and ($\models_{\mathsf{S}} \alpha_i \to \psi_2$ or $\models_{\mathsf{S}} \alpha_i \to \neg\psi_2$). There are thus four cases to consider. The first case is $\models_{\mathsf{S}} \alpha_i \to \psi_1$ and $\models_{\mathsf{S}} \alpha_i \to \psi_2$; in this case it follows that $\models_{\mathsf{S}} \alpha_i \to (\psi_1 \wedge \psi_2)$, i.e. $\models_{\mathsf{S}} \alpha_i \to \varphi$. The second case is $\models_{\mathsf{S}} \alpha_i \to \psi_1$ and $\models_{\mathsf{S}} \alpha_i \to \neg\psi_2$; in this case it follows that $\models_{\mathsf{S}} \alpha_i \to \neg(\psi_1 \wedge \psi_2)$, i.e. $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$. In the third and fourth case, we analogously arrive at $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$. □

Lemma 7 *For each $\varphi \in \mathbb{B}(\mathscr{F})$, it holds that $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid \models_{\mathsf{S}} \alpha_i \to \varphi\}$.*

Proof Consider an arbitrary S-model M; we will show that $M \models \varphi \Leftrightarrow M \models \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid \models_{\mathsf{S}} \alpha_i \to \varphi\}$.

($\Rightarrow$) Suppose $M \models \varphi$. By Lemma 3, there exists an $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$ such that $M \models \alpha_i$. Hence $M \not\models \alpha_i \to \neg\varphi$, and thus $\not\models_{\mathsf{S}} \alpha_i \to \neg\varphi$. By Lemma 6 it follows that $\models_{\mathsf{S}} \alpha_i \to \varphi$. Hence $M \models \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid \models_{\mathsf{S}} \alpha_i \to \varphi\}$.

($\Leftarrow$) Suppose $M \models \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid \models_{\mathsf{S}} \alpha_i \to \varphi\}$. Hence there is some $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$ such that $\models_{\mathsf{S}} \alpha_i \to \varphi$ and $M \models \alpha_i$. Hence $M \models \varphi$. □

3.2 Bitstring Semantics Based on Partitions

We are now in a position to define the bitstring semantics $\beta_{\mathsf{S}}^{\mathscr{F}}$ corresponding to a partition $\Pi_{\mathsf{S}}(\mathscr{F})$. It is important to stress that, although the partition $\Pi_{\mathsf{S}}(\mathscr{F})$ is induced by the fragment $\mathscr{F}$ itself, the corresponding bitstring semantics $\beta_{\mathsf{S}}^{\mathscr{F}}$ is defined for the fragment's entire Boolean closure $\mathbb{B}(\mathscr{F})$.

Definition 7 (bitstrings based on a partition) Consider a finite fragment $\mathscr{F}$ and the partition $\Pi_{\mathsf{S}}(\mathscr{F}) = \{\alpha_1, \ldots, \alpha_n\}$ induced by it. For every $\varphi \in \mathbb{B}(\mathscr{F})$, we define a bitstring $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi) \in \{0,1\}^n$ as follows:

$$\text{for each bit position } 1 \leq i \leq n\colon\ [\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i := \begin{cases} 1 & \text{if } \models_{\mathsf{S}} \alpha_i \to \varphi, \\ 0 & \text{if } \models_{\mathsf{S}} \alpha_i \to \neg\varphi. \end{cases}$$

Lemma 6 guarantees that the bitstring $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)$ is well-defined, in the sense that each of its bit positions gets assigned one and only one value. However, it should be emphasized that this guarantee only holds for formulas $\varphi \in \mathbb{B}(\mathscr{F})$: if $\varphi \notin \mathbb{B}(\mathscr{F})$, then it might happen that neither $\models_{\mathsf{S}} \alpha_i \to \varphi$ nor $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$, and thus $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i$ remains undefined. The alternative problem, viz. cases where $\models_{\mathsf{S}} \alpha_i \to \varphi$ and $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$ hold simultaneously, does not arise, since $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$ is, by Definition 5, S-consistent.[23] Furthermore, using Definition 7 we can now reformulate Lemma 7 as follows:

Lemma 8 *For each $\varphi \in \mathbb{B}(\mathscr{F})$, it holds that $\varphi \equiv_{\mathsf{S}} \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid [\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i = 1\}$.*

Each formula $\varphi \in \mathbb{B}(\mathscr{F})$ can thus be written as a disjunction of anchor formulas $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$, which are themselves conjunctions of (negated) formulas $\pm\varphi_j \in \mathscr{F}$. This illustrates the close conceptual connection between bitstrings and *disjunctive normal forms* (van Dalen, 2004, p. 25). The main difference seems to be that the latter are classically defined in terms of *propositional atoms* and their negations, whereas bitstrings involve *formulas from $\mathscr{F}$* and their negations.[24] Finally, note that Definition 7, Lemma 3 and the S-consistency of α_j together imply that $[\beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_i)]_j = 1 \Leftrightarrow \models_{\mathsf{S}} \alpha_j \to \alpha_i \Leftrightarrow i = j$ (for all $1 \leq i, j \leq n$), and $\beta_{\mathsf{S}}^{\mathscr{F}}$ thus maps the anchor formula α_i onto the unique bitstring which has 0 everywhere, except for a 1 in its i^{th} bit position:

Lemma 9 *For all anchor formulas $\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F})$, it holds that $\beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_i) = 0\cdots 0\underline{1}0\cdots 0$, with bit position i underlined.*

We are now able to state, prove and illustrate the paper's three key results.

Theorem 1 $\beta_{\mathsf{S}}^{\mathscr{F}} : \mathscr{F} \subseteq \mathbb{B}(\mathscr{F}) \to \{0,1\}^n$ *is a bitstring semantics for $\mathscr{F}$, and thus* $|\mathbb{B}(\mathscr{F})| = 2^n = 2^{|\Pi_{\mathsf{S}}(\mathscr{F})|}$.

Proof We first show that $\beta_{\mathsf{S}}^{\mathscr{F}}$ preserves Boolean structure. Consider arbitrary $\varphi, \psi \in \mathbb{B}(\mathscr{F})$. Since $\wedge$ is defined componentwise on $\{0,1\}^n$, it suffices to show that $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi \wedge \psi)]_i = [\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i \wedge [\beta_{\mathsf{S}}^{\mathscr{F}}(\psi)]_i$, or, equivalently: $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi \wedge \psi)]_i = 1$ iff $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i = 1$ and $[\beta_{\mathsf{S}}^{\mathscr{F}}(\psi)]_i = 1$ (for $1 \leq i \leq n$). By Definition 7, this reduces to showing that $\models_{\mathsf{S}} \alpha_i \to (\varphi \wedge \psi)$ iff $\models_{\mathsf{S}} \alpha_i \to \varphi$ and $\models_{\mathsf{S}} \alpha_i \to \psi$, which is straightforward. Similarly, showing that $[\beta_{\mathsf{S}}^{\mathscr{F}}(\neg\varphi)]_i = \neg[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i$ is equivalent to showing that $[\beta_{\mathsf{S}}^{\mathscr{F}}(\neg\varphi)]_i = 1$ iff $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i = 0$, and thus reduces by Definition 7 to the trivial claim that $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$ iff $\models_{\mathsf{S}} \alpha_i \to \neg\varphi$.

[23] In ongoing work, Demey (2017) provides a systematic analysis of the effects of $\beta_{\mathsf{S}}^{\mathscr{F}}$ outside the realm of $\mathbb{B}(\mathscr{F})$.

[24] This is also manifest in the induction on formula complexity in the proof of Lemma 6: the base case is not about propositional atoms, as usual, but rather about formulas from $\mathscr{F}$.

Next, we show that $\beta_{\mathsf{S}}^{\mathscr{F}} : \mathbb{B}(\mathscr{F}) \to \{0,1\}^n$ is a bijection. For surjectivity, note that for each bitstring $b \in \{0,1\}^n$ we can define the formula $\varphi_b := \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid [b]_i = 1\}$. Using Lemma 9 and the fact that $\beta_{\mathsf{S}}^{\mathscr{F}}$ preserves Boolean structure, we can check that $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi_b) = \beta_{\mathsf{S}}^{\mathscr{F}}(\bigvee_{[b]_i=1}\alpha_i) = \bigvee_{[b]_i=1}\beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_i) = b$.[25] For injectivity, consider formulas $\varphi, \psi \in \mathbb{B}(\mathscr{F})$ and suppose that $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi) = \beta_{\mathsf{S}}^{\mathscr{F}}(\psi)$; it now suffices to show that $\varphi \equiv_{\mathsf{S}} \psi$. Note that $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi) = \beta_{\mathsf{S}}^{\mathscr{F}}(\psi)$ means that $[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i = 1 \Leftrightarrow [\beta_{\mathsf{S}}^{\mathscr{F}}(\psi)]_i = 1$ for all $1 \leq i \leq n$, which together with Lemma 8 implies that $\varphi \equiv_{\mathsf{S}} \bigvee_{[\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi)]_i=1}\alpha_i = \bigvee_{[\beta_{\mathsf{S}}^{\mathscr{F}}(\psi)]_i=1}\alpha_i \equiv_{\mathsf{S}} \psi$. □

Example 3 Consider again the FOL-fragment $\mathscr{F}^\dagger = \{\forall x Px,\ \exists x Px,\ \neg Pa\}$. In Example 1 we showed that this fragment induces a partition $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$ with anchor formulas $\alpha_1 := \forall x Px$, $\alpha_2 := Pa \wedge \neg\forall x Px$, $\alpha_3 := \neg Pa \wedge \exists x Px$ and $\alpha_4 := \neg\exists x Px$. Based on this partition, the formula $\exists x Px \in \mathscr{F}^\dagger$ gets assigned a bitstring $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px)$ as follows:

$$\begin{array}{lrcrll} \models_{\mathsf{FOL}} & \forall x Px & \to & \exists x Px & \text{and thus} & [\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px)]_1 = 1 \\ \models_{\mathsf{FOL}} & (Pa \wedge \neg\forall x Px) & \to & \exists x Px & \text{and thus} & [\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px)]_2 = 1 \\ \models_{\mathsf{FOL}} & (\neg Pa \wedge \exists x Px) & \to & \exists x Px & \text{and thus} & [\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px)]_3 = 1 \\ \models_{\mathsf{FOL}} & \neg\exists x Px & \to & \neg\exists x Px & \text{and thus} & [\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px)]_4 = 0 \end{array}$$

In sum, we find that $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px) = 1110$. The remaining two formulas in $\mathscr{F}^\dagger$ are mapped onto the following bitstrings: $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\forall x Px) = 1000$ and $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\neg Pa) = 0011$. The formula $\neg Pa \wedge \exists x Px$ is not in $\mathscr{F}^\dagger$, but it does belong to its Boolean closure $\mathbb{B}(\mathscr{F}^\dagger)$, and can thus also be assigned a bitstring: $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\neg Pa \wedge \exists x Px) = 0010$. Finally, note that $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\neg Pa) \wedge \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px) = 0011 \wedge 1110 = 0010 = \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\neg Pa \wedge \exists x Px)$, illustrating the fact that $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$ preserves Boolean structure.

Example 4 Consider again the CPL-fragment $\mathscr{F}^\ddagger = \{p \wedge q,\ \neg p \vee \neg q,\ p \vee q,\ \neg p \wedge \neg q\}$. In Example 2 we showed that this fragment induces a partition $\Pi_{\mathsf{CPL}}(\mathscr{F}^\ddagger)$ with anchor formulas $\alpha_1 := p \wedge q$, $\alpha_2 := (p \vee q) \wedge (\neg p \vee \neg q)$ and $\alpha_3 := \neg p \wedge \neg q$. The bitstring semantics $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}$ corresponding to this partition maps the formulas of $\mathscr{F}^\ddagger$ onto the following bitstrings: $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}(p \wedge q) = 100$, $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}(\neg p \vee \neg q) = 011$, $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}(p \vee q) = 110$ and $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}(\neg p \wedge \neg q) = 001$. As to the Boolean closure $\mathbb{B}(\mathscr{F}^\ddagger)$, we find that $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}((p \vee q) \wedge (\neg p \vee \neg q)) = 010$ and $\beta_{\mathsf{CPL}}^{\mathscr{F}^\ddagger}(p \leftrightarrow q) = 101$.

We have just seen that for any fragment $\mathscr{F}$ of formulas from some logical system S, the mapping $\beta_{\mathsf{S}}^{\mathscr{F}}$ is a bitstring semantics, i.e. a Boolean isomorphism. Since Aristotelian structure is fully determined by Boolean structure, this mapping is also an Aristotelian isomorphism (recall Definition 4).

Theorem 2 *$\beta_{\mathsf{S}}^{\mathscr{F}} : \mathscr{F} \subseteq \mathbb{B}(\mathscr{F}) \to \{0,1\}^n$ is an Aristotelian isomorphism.*

Proof This follows immediately from Theorem 1 and Lemma 2. □

[25] For example, for $b = 1010 \in \{0,1\}^4$ we define $\varphi_{1010} := \bigvee\{\alpha_i \in \Pi_{\mathsf{S}}(\mathscr{F}) \mid [1010]_i = 1\} = \alpha_1 \vee \alpha_3$, and thus $\beta_{\mathsf{S}}^{\mathscr{F}}(\varphi_{1010}) = \beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_1 \vee \alpha_3) = \beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_1) \vee \beta_{\mathsf{S}}^{\mathscr{F}}(\alpha_3) = 1000 \vee 0010 = 1010 = b$.

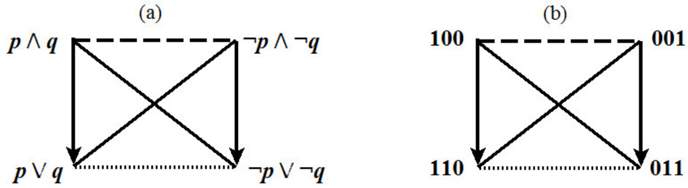

Fig. 3 (a) Aristotelian square for $\mathscr{F}^\ddagger$, (b) Aristotelian square for $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}[\mathscr{F}^\ddagger]$.

Example 5 Consider again the FOL-fragment $\mathscr{F}^\dagger$ studied in Examples 1 and 3, where it was shown that $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\exists xPx) = 1110$, $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\forall xPx) = 1000$ and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\neg Pa) = 0011$. As to the Aristotelian relation between $\forall xPx$ and $\neg Pa$, we find that $\models_{\mathsf{FOL}} \neg(\forall xPx \wedge \neg Pa)$ and $\not\models_{\mathsf{FOL}} \forall xPx \vee \neg Pa$, and thus $\forall xPx$ and $\neg Pa$ are FOL-contrary (Definition 1). Furthermore, since $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\forall xPx) \wedge \beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\neg Pa) = 1000 \wedge 0011 = 0000$ and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\forall xPx) \vee \beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\neg Pa) = 1000 \vee 0011 = 1011 \neq 1111$, we find that $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\forall xPx)$ and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\neg Pa)$ are 4-contrary (Definition 2). Completely analogously, we find that the formulas $\exists xPx$ and $\neg Pa$ are FOL-subcontrary and that their bitstring representations $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\exists xPx)$ and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\neg Pa)$ are 4-subcontrary, while $\forall xPx$ and $\exists xPx$ are in FOL-subalternation and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\forall xPx)$ and $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}(\exists xPx)$ are in 4-subalternation. This illustrates the fact that the Boolean isomorphism $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}$ is also an Aristotelian isomorphism.

Example 6 Consider again the CPL-fragment $\mathscr{F}^\ddagger$ studied in Examples 2 and 4, where it was shown that $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(p \wedge q) = 100$, $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \vee \neg q) = 011$, $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(p \vee q) = 110$ and $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \wedge \neg q) = 001$. The Aristotelian relations holding between the formulas of $\mathscr{F}^\ddagger$, and between their bitstring counterparts, are:

$$
\begin{array}{rr}
CD_{\mathsf{CPL}}(\ \ p \wedge \ \ q, \neg p \vee \neg q) & CD_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \wedge \ \ q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \vee \neg q)) \\
CD_{\mathsf{CPL}}(\ \ p \vee \ \ q, \neg p \wedge \neg q) & CD_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \vee \ \ q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \wedge \neg q)) \\
C_{\mathsf{CPL}}(\ \ p \wedge \ \ q, \neg p \wedge \neg q) & C_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \wedge \ \ q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \wedge \neg q)) \\
SC_{\mathsf{CPL}}(\ \ p \vee \ \ q, \neg p \vee \neg q) & SC_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \vee \ \ q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \vee \neg q)) \\
SA_{\mathsf{CPL}}(\ \ p \wedge \ \ q, \ \ p \vee \ \ q) & SA_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \wedge \ \ q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\ \ p \vee \ \ q)) \\
SA_{\mathsf{CPL}}(\neg p \wedge \neg q, \neg p \vee \neg q) & SA_3(\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \wedge \neg q), \beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}(\neg p \vee \neg q))
\end{array}
$$

The formulas of $\mathscr{F}^\ddagger$ and the bitstrings of $\beta^{\mathscr{F}^\ddagger}_{\mathsf{CPL}}[\mathscr{F}^\ddagger]$ thus constitute Aristotelian squares, which are shown in Figure 3(a) and (b), respectively.

For any fragment $\mathscr{F}$ of formulas from some logical system S, the Boolean (and Aristotelian) isomorphism $\beta^{\mathscr{F}}_{\mathsf{S}}$ assigns bitstrings to the formulas in $\mathscr{F}$. Furthermore, the assigned bitstrings are of *minimal length*: they are the shortest ones that still adequately capture the Boolean structure of $\mathscr{F}$.

Theorem 3 $\beta_{\mathsf{S}}^{\mathscr{F}}: \mathscr{F} \subseteq \mathbb{B}(\mathscr{F}) \to \{0,1\}^n$ *is a minimal bitstring semantics for* $\mathscr{F}$*, i.e. every bitstring semantics* $\beta: \mathscr{F} \subseteq \mathbb{B}_k \to \{0,1\}^k$ *of* $\mathscr{F}$ *that is not a permutation variant of* $\beta_{\mathsf{S}}^{\mathscr{F}}$ *makes use of bitstrings of length* $k > n$*.*[26]

Proof This follows immediately from Theorem 1 and the fact that $\mathbb{B}(\mathscr{F})$ is, by Definition 3, the *smallest* Boolean algebra that contains $\mathscr{F}$. □

Note that $\beta_{\mathsf{S}}^{\mathscr{F}}$ assigns bitstrings not just to (the formulas in) $\mathscr{F}$ itself, but to its entire Boolean closure $\mathbb{B}(\mathscr{F})$, and thus also to every other fragment $\mathscr{F}' \subseteq \mathbb{B}(\mathscr{F})$. Since $\mathbb{B}(\mathscr{F})$ is a Boolean algebra that contains $\mathscr{F}'$, it follows that $\beta_{\mathsf{S}}^{\mathscr{F}}: \mathscr{F}' \subseteq \mathbb{B}(\mathscr{F}) \to \{0,1\}^n$ is also a bitstring semantics for the fragment $\mathscr{F}'$. Furthermore, if $\mathbb{B}(\mathscr{F}') = \mathbb{B}(\mathscr{F})$, then $\beta_{\mathsf{S}}^{\mathscr{F}}$ is even a *minimal* bitstring semantics for $\mathscr{F}'$ (in the sense of Theorem 3); however, if $\mathbb{B}(\mathscr{F}') \subsetneq \mathbb{B}(\mathscr{F})$, then $\beta_{\mathsf{S}}^{\mathscr{F}}$ is a *not* a minimal semantics for $\mathscr{F}'$. In the latter case, the technique described above can still be used to define another bitstring semantics $\beta_{\mathsf{S}}^{\mathscr{F}'}$, which *is* minimal for $\mathscr{F}'$.

Example 7 We return, one final time, to the fragment $\mathscr{F}^\dagger = \{\forall x Px, \neg Pa, \exists x Px\}$ that was studied in Examples 1, 3 and 5. In Example 1 it was shown that $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$ consists of 4 anchor formulas, and thus every formula $\varphi \in \mathbb{B}(\mathscr{F}^\dagger)$ can be represented as a bitstring $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\varphi)$ of length 4, as was illustrated in Example 3. Consider the two new fragments $\mathscr{F}_*^\dagger := \{\exists x \neg Px, \neg Pa, \exists x Px\}$ and $\mathscr{F}_{**}^\dagger := \{\forall x Px, \exists x Px, \forall x \neg Px\}$. Since $\mathscr{F}_*^\dagger, \mathscr{F}_{**}^\dagger \subseteq \mathbb{B}(\mathscr{F}^\dagger)$, both of these fragments can be assigned bitstrings by means of $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$:

$$\begin{array}{lcllcl}
\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x \neg Px) & = & 0111, & \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\forall x Px) & = & 1000, \\
\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\neg Pa) & = & 0011, & \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px) & = & 1110, \\
\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\exists x Px) & = & 1110, & \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}(\forall x \neg Px) & = & 0001.
\end{array}$$

Hence, $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$ is not only a bitstring semantics for $\mathscr{F}^\dagger$, but also for $\mathscr{F}_*^\dagger$ and $\mathscr{F}_{**}^\dagger$. One can show that $\mathbb{B}(\mathscr{F}_*^\dagger) = \mathbb{B}(\mathscr{F}^\dagger)$, and thus $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$ is even a *minimal* bitstring semantics for $\mathscr{F}_*^\dagger$. However, since $\mathbb{B}(\mathscr{F}_{**}^\dagger) \subsetneq \mathbb{B}(\mathscr{F}^\dagger)$,[27] it follows that $\beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$ is *not* a minimal semantics for $\mathscr{F}_{**}^\dagger$. To obtain a semantics for $\mathscr{F}_{**}^\dagger$ that does exhibit minimality, we calculate $\Pi_{\mathsf{FOL}}(\mathscr{F}_{**}^\dagger) = \{\forall x Px, \exists x Px \wedge \exists x \neg Px, \forall x \neg Px\}$,[28] and thus obtain a new bitstring semantics $\beta_{\mathsf{FOL}}^{\mathscr{F}_{**}^\dagger}$ that works with bitstrings of length 3 (instead of length 4). In particular, we see that $\beta_{\mathsf{FOL}}^{\mathscr{F}_{**}^\dagger}(\forall x Px) = 100, \beta_{\mathsf{FOL}}^{\mathscr{F}_{**}^\dagger}(\exists x Px) = 110$ and $\beta_{\mathsf{FOL}}^{\mathscr{F}_{**}^\dagger}(\forall x \neg Px) = 001$. Notice, incidentally, that we can analogously construct $\beta_{\mathsf{FOL}}^{\mathscr{F}_*^\dagger}$, but this is redundant, since $\mathbb{B}(\mathscr{F}_*^\dagger) = \mathbb{B}(\mathscr{F}^\dagger)$ entails that $\beta_{\mathsf{FOL}}^{\mathscr{F}_*^\dagger} = \beta_{\mathsf{FOL}}^{\mathscr{F}^\dagger}$.

[26] We say that β is a permutation variant of $\beta_{\mathsf{S}}^{\mathscr{F}}$ iff there is a permutation $\pi: \{0,1\}^n \to \{0,1\}^n$ such that $\beta = \pi \circ \beta_{\mathsf{S}}^{\mathscr{F}}$; recall Footnote 19.

[27] For example, note that $Pa \wedge \exists x \neg Px \in \mathbb{B}(\mathscr{F}^\dagger) - \mathbb{B}(\mathscr{F}_{**}^\dagger)$.

[28] Comparing the new partition $\Pi_{\mathsf{FOL}}(\mathscr{F}_{**}^\dagger)$ with the original partition $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$, we see that the two original anchor formulas $\alpha_2, \alpha_3 \in \Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$, i.e. $Pa \wedge \neg \forall x Px$ and $\neg Pa \wedge \exists x Px$, are collapsed into a single new anchor formula, viz. $(Pa \wedge \neg \forall x Px) \vee (\neg Pa \wedge \exists x Px) \equiv_{\mathsf{FOL}} \exists x Px \wedge \exists x \neg Px \in \Pi_{\mathsf{FOL}}(\mathscr{F}_{**}^\dagger)$.

Example 8 Finally, we also return one final time to the fragment $\mathscr{F}^{\ddagger} = \{p \wedge q, \neg p \vee \neg q, p \vee q, \neg p \wedge \neg q\}$ that was studied in Examples 2, 4 and 6. There it was shown that the minimal bitstring semantics $\beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}}$ for this fragment works with bitstrings of length 3. Based on the truth tables for binary propositional connectives, however, it seems more intuitive to represent $\mathscr{F}^{\ddagger}$ with bitstrings of length 4 instead of length 3. This suggests that the minimal semantics for a given fragment is not always the most 'natural' one. A more appropriate semantics for $\mathscr{F}^{\ddagger}$ can easily be obtained, by defining a new fragment $\mathscr{F}^{\ddagger}_{*} := \{p, q\}$. The partition induced by this fragment is $\Pi_{\mathsf{CPL}}(\mathscr{F}^{\ddagger}_{*}) = \{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$,[29] and thus the corresponding bitstring semantics $\beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}$ works with bitstrings of length 4, and hence $\mathbb{B}(\mathscr{F}^{\ddagger}) \subsetneq \mathbb{B}(\mathscr{F}^{\ddagger}_{*})$. Since $\mathscr{F}^{\ddagger} \subseteq \mathbb{B}(\mathscr{F}^{\ddagger}_{*})$, it can be represented by means of $\beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}$. In particular, we find that $\beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}(p \wedge q) = 1000, \beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}(\neg p \vee \neg q) = 0111, \beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}(p \vee q) = 1110$ and $\beta_{\mathsf{CPL}}^{\mathscr{F}^{\ddagger}_{*}}(\neg p \wedge \neg q) = 0001$—thus essentially capturing the truth tables of the propositional connectives in these formulas.

3.3 Correlation between Fragment Size and Bitstring Length

We will now investigate the correlation between the size of (i.e. the number of formulas in) a given logical fragment on the one hand, and the minimal bitstring length (i.e. the minimal number of bit positions) that is required to represent it on the other hand. This correlation is not strictly deterministic—specifying a fragment size does not uniquely determine minimal bitstring length (nor vice versa)—, and hence, the best we can hope for is not to obtain a *unique* minimal bitstring length corresponding to a given fragment size (or vice versa), but rather to obtain a *range* of possible values. Theorems 4 and 5 will do exactly this, by providing tight lower and upper bounds on fragment size and minimal bitstring length, respectively. The key insight is that by Definition 7 and Theorem 3, the minimal[30] number of bit positions required to represent a fragment $\mathscr{F}$ is exactly the number of cells in the partition induced by that fragment: $|\Pi_{\mathsf{S}}(\mathscr{F})|$.

The first question concerns the size of a fragment, given that the minimal bitstring length needed to represent it is some given number n. Obtaining a tight upper bound on the fragment size is nearly trivial, but finding a tight lower bound is significantly more involved:[31]

Theorem 4 *For a logical fragment $\mathscr{F}$ such that $|\Pi_{\mathsf{S}}(\mathscr{F})| = n \geq 2$, it holds that $\lceil \log_2(n) \rceil \leq |\mathscr{F}| \leq 2^n$. Furthermore, these bounds are tight, i.e. there exist frag-*

[29] Comparing the new partition $\Pi_{\mathsf{CPL}}(\mathscr{F}^{\ddagger}_{*})$ with the original partition $\Pi_{\mathsf{CPL}}(\mathscr{F}^{\ddagger})$, we see that the original anchor formula $\alpha_2 \in \Pi_{\mathsf{CPL}}(\mathscr{F}^{\ddagger})$, i.e. $(p \vee q) \wedge (\neg p \vee \neg q)$, has been split into two new anchor formulas in $\Pi_{\mathsf{CPL}}(\mathscr{F}^{\ddagger}_{*})$, viz. $p \wedge \neg q$ and $\neg p \wedge q$, in the sense that $(p \vee q) \wedge (\neg p \vee \neg q) \equiv_{\mathsf{CPL}} (p \wedge \neg q) \vee (\neg p \wedge q)$.

[30] Note that we are only interested in the *minimal* number of bit positions (i.e. the *minimal* bitstring length) that is required to represent a fragment of a given size. After all, if a fragment can be represented by bitstrings of length n, then it can trivially also be represented by bitstrings of length k, for any $k \geq n$.

[31] Theorem 4 assumes that $|\Pi_{\mathsf{S}}(\mathscr{F})| \geq 2$. For the sake of completeness, note that if $|\Pi_{\mathsf{S}}(\mathscr{F})| = 1$, then $1 \leq |\mathscr{F}| \leq 2$. To see this, note that $|\Pi_{\mathsf{S}}(\mathscr{F})| = 1$ means that $\Pi_{\mathsf{S}}(\mathscr{F}) = \{\top\}$, and thus $\mathscr{F} = \{\top\}$ or $\mathscr{F} = \{\bot\}$ or $\mathscr{F} = \{\top, \bot\}$.

ments $\mathscr{F}_L$ and $\mathscr{F}_U$ such that $|\Pi_{\mathsf{S}}(\mathscr{F}_L)| = n = |\Pi_{\mathsf{S}}(\mathscr{F}_U)|$, and $|\mathscr{F}_L| = \lceil \log_2(n) \rceil$ and $|\mathscr{F}_U| = 2^n$.

Proof For the upper bound, note that there are exactly 2^n bitstrings of length n, and hence we can represent 2^n formulas using these bitstrings, i.e. $|\mathscr{F}| \leq 2^n$. For tightness, take $\mathscr{F}_U := \mathbb{B}_n$.

For the lower bound, first note that since $n \geq 2$, there exists a number $k \in \mathbb{N}$ such that $2^k < n \leq 2^{k+1}$, and thus $k = \log_2(2^k) < \log_2(n) \leq \log_2(2^{k+1}) = k+1$, i.e. $k + 1 = \lceil \log_2(n) \rceil$. Now note that $\lceil \log_2(n) \rceil \leq |\mathscr{F}|$ iff $k+1 \leq |\mathscr{F}|$ iff $k < |\mathscr{F}|$. It thus suffices to show that $k < |\mathscr{F}|$. Toward a contradiction, suppose that $k \geq |\mathscr{F}|$. Note that the anchor formulas in $\Pi_{\mathsf{S}}(\mathscr{F})$ are S-consistent conjunctions of the form $\pm\varphi_1 \wedge \cdots \wedge \pm\varphi_{|\mathscr{F}|}$; since there are exactly $2^{|\mathscr{F}|}$ conjunctions of this form, it follows that $|\Pi_{\mathsf{S}}(\mathscr{F})| \leq 2^{|\mathscr{F}|}$. Putting everything together, we find that $n = |\Pi_{\mathsf{S}}(\mathscr{F})| \leq 2^{|\mathscr{F}|} \leq 2^k < n$, which is a contradiction.

For tightness of the lower bound, we will work in CPL. Consider again the number k such that $2^k < n \leq 2^{k+1}$, and thus $k+1 = \lceil \log_2(n) \rceil$. First define the auxiliary fragment $\mathscr{F}^\circ := \{p_1, \ldots, p_k\}$. Since the anchor formulas in $\Pi_{\mathsf{S}}(\mathscr{F}^\circ)$ are S-consistent conjunctions of the form $\alpha_i = \pm p_1 \wedge \cdots \wedge \pm p_k$ and there are exactly 2^k such S-consistent conjunctions, we can write $\Pi_{\mathsf{CPL}}(\mathscr{F}^\circ) = \{\alpha_1, \ldots, \alpha_{2^k}\}$. Since $2^k < n$, we know that $n - 2^k > 0$ and thus $n - 2^k \geq 1$. Furthermore, since $n \leq 2^{k+1} = 2 \cdot 2^k = 2^k + 2^k$, it follows that $n - 2^k \leq 2^k$. Putting $b := n - 2^k$, we thus have $1 \leq b \leq 2^k$. Now define the formula $\varphi := p_{k+1} \vee \bigvee_{j=b+1}^{j=2^k} \alpha_j$, and define $\mathscr{F}_L := \mathscr{F}^\circ \cup \{\varphi\}$. It is easy to see that $\varphi \not\equiv_{\mathsf{CPL}} p_i$ for all $p_i \in \mathscr{F}^\circ$, and thus $|\mathscr{F}_L| = |\mathscr{F}^\circ \cup \{\varphi\}| = |\mathscr{F}^\circ| + 1 = k+1 = \lceil \log_2(n) \rceil$.

Finally, we show that $|\Pi_{\mathsf{CPL}}(\mathscr{F}_L)| = n$. It is easy to see that φ is CPL-contingent, and hence $\Pi_{\mathsf{CPL}}(\{\varphi\}) = \{\varphi, \neg\varphi\}$. It follows by Lemma 4 that $\Pi_{\mathsf{CPL}}(\mathscr{F}_L) = \Pi_{\mathsf{CPL}}(\mathscr{F}^\circ \cup \{\varphi\}) = \Pi_{\mathsf{CPL}}(\mathscr{F}_L) \wedge_{\mathsf{CPL}} \Pi_{\mathsf{CPL}}(\{\varphi\}) = \{\alpha_1, \ldots, \alpha_{2^k}\} \wedge_{\mathsf{CPL}} \{\varphi, \neg\varphi\}$; the anchor formulas in $\Pi_{\mathsf{CPL}}(\mathscr{F}_L)$ are thus exactly the CPL-consistent conjunctions $\alpha_i \wedge \varphi$ and $\alpha_i \wedge \neg\varphi$, with $1 \leq i \leq 2^k$. Using Lemma 3, we see that

- for $1 \leq i \leq b$: $\alpha_i \wedge \varphi \equiv_{\mathsf{CPL}} \alpha_i \wedge \left(p_{k+1} \vee \bigvee_{j=b+1}^{j=2^k} \alpha_j\right) \equiv_{\mathsf{CPL}}$ $(\alpha_i \wedge p_{k+1}) \vee \bigvee_{j=b+1}^{j=2^k} (\alpha_i \wedge \alpha_j) \equiv_{\mathsf{CPL}} \alpha_i \wedge p_{k+1}$,
- for $1 \leq i \leq b$: $\alpha_i \wedge \neg\varphi \equiv_{\mathsf{CPL}} \alpha_i \wedge \neg\left(p_{k+1} \vee \bigvee_{j=b+1}^{j=2^k} \alpha_j\right) \equiv_{\mathsf{CPL}}$ $\alpha_i \wedge \neg p_{k+1} \wedge \bigwedge_{j=b+1}^{j=2^k} \neg\alpha_j \equiv_{\mathsf{CPL}} \alpha_i \wedge \neg p_{k+1}$,
- for $b+1 \leq i \leq 2^k$: $\alpha_i \wedge \varphi \equiv_{\mathsf{CPL}} \alpha_i \wedge \left(p_{k+1} \vee \bigvee_{j=b+1}^{j=2^k} \alpha_j\right) \equiv_{\mathsf{CPL}} \alpha_i$,
- for $b+1 \leq i \leq 2^k$: $\alpha_i \wedge \neg\varphi \equiv_{\mathsf{CPL}} \alpha_i \wedge \neg\left(p_{k+1} \vee \bigvee_{j=b+1}^{j=2^k} \alpha_j\right) \equiv_{\mathsf{CPL}}$ $\alpha_i \wedge \neg p_{k+1} \wedge \bigwedge_{j=b+1}^{j=2^k} \neg\alpha_j \equiv_{\mathsf{CPL}} \bot$.

Hence $\Pi_{\mathsf{CPL}}(\mathscr{F}_L) = \{\alpha_1 \wedge p_{k+1}, \alpha_1 \wedge \neg p_{k+1}, \ldots, \alpha_b \wedge p_{k+1}, \alpha_b \wedge \neg p_{k+1}, \alpha_{b+1}, \ldots, \alpha_{2^k}\}$ (i.e. the first b anchor formulas of $\Pi_{\mathsf{CPL}}(\mathscr{F}^\circ)$ have been 'split' into two), and thus $|\Pi_{\mathsf{CPL}}(\mathscr{F}_L)| = 2b + (2^k - b) = b + 2^k = (n - 2^k) + 2^k = n$. $\square$

The second question is basically the inverse of the first, and concerns how many bit positions are minimally required to represent a logical fragment of a given size.

In general, larger fragments require bitstrings of higher lengths: Lemma 5 entails that if $\mathscr{F}_1 \subseteq \mathscr{F}_2$, then $|\Pi_{\mathsf{S}}(\mathscr{F}_1)| \leq |\Pi_{\mathsf{S}}(\mathscr{F}_2)|$. However, the minimal bitstring length required to represent a fragment not only depends on the fragment's size, but also on its Aristotelian and even its Boolean structure. Hence, in general (i.e. without knowing the details of the fragment's Boolean structure) we can only establish a range of possible minimal bitstring lengths:[32]

Theorem 5 *For a logical fragment $\mathscr{F}$ of size $m := |\mathscr{F}| \geq 2$, it holds that $\lceil \log_2(m) \rceil \leq |\Pi_{\mathsf{S}}(\mathscr{F})| \leq 2^m$. Furthermore, these bounds are tight, i.e. there exist fragments $\mathscr{F}_L$ and $\mathscr{F}_U$ of size $|\mathscr{F}_L| = m = |\mathscr{F}_U|$ such that $|\Pi_{\mathsf{S}}(\mathscr{F}_L)| = \lceil \log_2(m) \rceil$ and $|\Pi_{\mathsf{S}}(\mathscr{F}_U)| = 2^m$.*

Proof For the upper bound, recall that the formulas in $\Pi_{\mathsf{S}}(\mathscr{F})$ are S-consistent conjunctions of the form $\pm\varphi_1 \wedge \cdots \wedge \pm\varphi_m$ (where $\mathscr{F} = \{\varphi_1, \ldots, \varphi_m\}$); there are exactly 2^m conjunctions of this form. For tightness, note that $|\Pi_{\mathsf{S}}(\mathscr{F})| = 2^m$ means that *all* of these 2^m conjunctions are S-consistent, which happens, for example, in CPL if we take $\mathscr{F}_U := \{p_1, \ldots, p_m\}$.

For the lower bound, first note that since $m \geq 2$, there exists a number $k \in \mathbb{N}$ such that $2^k < m \leq 2^{k+1}$, and thus $k = \log_2(2^k) < \log_2(m) \leq \log_2(2^{k+1}) = k+1$, i.e. $k+1 = \lceil \log_2(m) \rceil$. Toward a contradiction, suppose that $\lceil \log_2(m) \rceil > |\Pi_{\mathsf{S}}(\mathscr{F})|$. This means that $|\Pi_{\mathsf{S}}(\mathscr{F})| < k+1$, and thus $|\Pi_{\mathsf{S}}(\mathscr{F})| \leq k$. By Theorem 4, it follows that $m = |\mathscr{F}| \leq 2^{|\Pi_{\mathsf{S}}(\mathscr{F})|} \leq 2^k$, which contradicts $2^k < m$.

For tightness of the lower bound, let $\mathbb{B}_{k+1}$ be a Boolean algebra of 2^{k+1} S-formulas, and choose some fragment $\mathscr{F}_L \subseteq \mathbb{B}_{k+1}$ such that $|\mathscr{F}_L| = m$ (this is certainly possible since $|\mathscr{F}_L| = m \leq 2^{k+1} = |\mathbb{B}_{k+1}|$). Since $2^k < m \leq 2^{k+1}$, it follows that $\mathbb{B}_{k+1}$ is the smallest Boolean algebra that contains $\mathscr{F}_L$, i.e. $\mathbb{B}(\mathscr{F}_L) = \mathbb{B}_{k+1}$, and hence $|\Pi_{\mathsf{S}}(\mathscr{F}_L)| = k+1 = \lceil \log_2(m) \rceil$. $\square$

If we have a logical fragment $\mathscr{F}$ of size $m := |\mathscr{F}|$ and the partition induced by it is of size $n := |\Pi_{\mathsf{S}}(\mathscr{F})|$, then

- Theorem 4 bounds m in terms of n: $\lceil \log_2(n) \rceil \leq m \leq 2^n$,
- Theorem 5 bounds n in terms of m: $\lceil \log_2(m) \rceil \leq n \leq 2^m$.

In this sense, Theorems 4 and 5 can be said to be each other's inverses. The lower and upper bounds in these theorems are resp. logarithmic and exponential, and thus diverge at a double-exponential rate. Finally, it should be emphasized that these theorems hold for *arbitrary* fragments. However, if the fragments are known to satisfy certain conditions, then further improvements can be made on the bounds. In many applications in logical geometry, for example, bitstrings are used to study Aristotelian diagrams, and then, the fragments typically satisfy two conditions: (i) they only contain S-contingent formulas ($\bot, \top \notin \mathscr{F}$), and (ii) they are closed under negation (if $\varphi \in \mathscr{F}$, then there is some $\psi \in \mathscr{F}$ such that $\psi \equiv \neg\varphi$). For fragments satisfying these two conditions, the proofs of Theorems 4 and 5 can be adapted to yield new and sharper

[32] Theorem 5 assumes that $|\mathscr{F}| \geq 2$. For the sake of completeness, note that if $|\mathscr{F}| = 1$, then $1 \leq |\Pi_{\mathsf{S}}(\mathscr{F})| \leq 2$. To see this, suppose that $\mathscr{F} = \{\varphi\}$; if φ is S-contingent, then $\Pi_{\mathsf{S}}(\mathscr{F}) = \{\varphi, \neg\varphi\}$, otherwise $\Pi_{\mathsf{S}}(\mathscr{F}) = \{\top\}$.

- bounds for m in terms of n: $2\lceil \log_2(n) \rceil \leq m \leq 2^n - 2$,
- bounds for n in terms of m: $\lceil \log_2(m+2) \rceil \leq n \leq 2^{\frac{m}{2}}$.

In comparison to the original theorems, we see that all bounds have improved, but to widely varying degrees. The lower bound for bitstring length n and the upper bound for fragment size m hardly improve at all; in contrast, the lower bound for m doubles, and the upper bound for n even decreases exponentially.

4 Dealing with Logic-Sensitivity

In the remainder of this paper, we will explain and illustrate the usefulness of the bitstring technique developed in Section 3, by showing how it avoids the problems of the original, more informal approach that were described in Subsection 2.3. Recall that the first problem concerned the relative (in)sensitivity of bitstrings with respect to the specific properties of the underlying logical system: the Aristotelian relation holding between two formulas may depend on the underlying logical system, but the Aristotelian relation holding between two bitstrings is uniquely determined by those bitstrings.

Without a doubt, the most widely known example of the 'logic-sensitivity' of the Aristotelian relations is the problem of *existential import* in the classical square of oppositions.[33] Consider the following four formulas, which are the usual first-order translations of the categorical statements from syllogistics:

- $\forall x(Sx \rightarrow Px)$ (traditionally called the A-statement),
- $\exists x(Sx \wedge Px)$ (traditionally called the I-statement),
- $\forall x(Sx \rightarrow \neg Px)$ (traditionally called the E-statement),
- $\exists x(Sx \wedge \neg Px)$ (traditionally called the O-statement).

Both in classical syllogistics (SYL) and in contemporary first-order logic (FOL), the A- and O-statements and the E- and I-statements are contradictory to each other. Whether there are any other Aristotelian relations among these formulas, however, depends on the underlying logical system. For example, since SYL has the existential import axiom $\exists xSx$,[34] it can be shown that $\models_{\mathsf{SYL}} \neg[\forall x(Sx \rightarrow Px) \ \wedge \ \forall x(Sx \rightarrow \neg Px)]$ and that $\not\models_{\mathsf{SYL}} \forall x(Sx \rightarrow Px) \ \vee \ \forall x(Sx \rightarrow \neg Px)$, and thus the A- and E-statements are SYL-contrary. In a similar fashion, it follows that the I- and O-statements are SYL-subcontrary, and that there are SYL-subalternations from A to I and from E to O. In sum, then, in SYL these four statements yield a classical Aristotelian square,

33 Demey and Smessaert (2016, Subsection 5.2) and Demey (2015) systematically study the logic-sensitivity of some other well-known Aristotelian diagrams, and its effects on the bitstrings that are used to represent those diagrams.

34 The existential import of syllogistics is here formalized by including $\exists xSx$ as an axiom (so we have $\mathsf{SYL} = \mathsf{FOL} \cup \{\exists xSx\}$). Another formalization involves adding $\exists xSx$ as a conjunct to the categorical statements: for $\varphi \in \{\mathrm{A}, \mathrm{I}, \mathrm{E}, \mathrm{O}\}$, put $\varphi_{\mathrm{imp!}} := \varphi \wedge \exists xSx$ (Chatti and Schang, 2013, Definition 4), but then more needs to be said about the contradictions—for example, although A and O are contradictory to each other, $\mathrm{A}_{\mathrm{imp!}} = \exists xSx \wedge \forall x(Sx \rightarrow Px)$ is not contradictory to $\mathrm{O}_{\mathrm{imp!}} = \exists xSx \wedge \exists x(Sx \wedge \neg Px)$, but rather to $\neg\exists xSx \vee \exists x(Sx \wedge \neg Px)$, which is equivalent to Chatti and Schang's (2013, Definition 5) $\mathrm{O}_{\mathrm{imp?}}$.

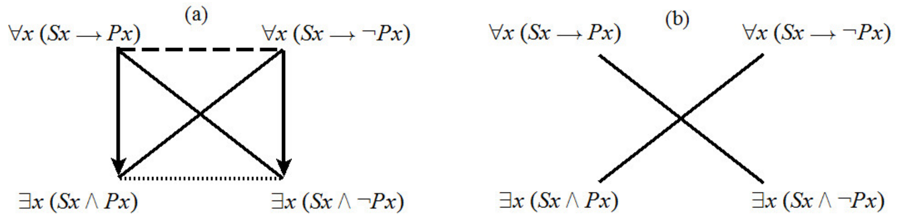

Fig. 4 Aristotelian squares for the categorical statements (a) in syllogistics (SYL) and (b) in first-order logic (FOL).

which is shown in Figure 4(a). By contrast, in FOL we do not have $\exists x Sx$ as an axiom, and as a consequence there are no Aristotelian relations besides the two contradictions (A/O and I/E). For example, the A- and E-statements are no longer contrary to each other, since there exist FOL-models $M = \langle D, I \rangle$ such that $I(S) = \emptyset$, and thus $M \models \forall x(Sx \to Px) \wedge \forall x(Sx \to \neg Px)$. Diagrammatically speaking, by going from SYL to FOL, we observe that "the square of opposition became an X of opposition" (Béziau and Payette, 2012, p. 13), which is shown in Figure 4(b).[35] Note that although the two Aristotelian diagrams in Figure 4 contain the same four formulas, they have different constellations of Aristotelian relations, i.e. they are not Aristotelian isomorphic (Definition 4), and thus, a fortiori, not Boolean isomorphic (Lemma 1).[36]

We will now discuss how the logic-sensitivity of this Aristotelian diagram can be captured in a precise and systematic way by the bitstring approach that was developed in Section 3. We begin by considering the fragment $\mathscr{F}$ that consists of the A- and E-statements:[37]

$$\mathscr{F} := \{\forall x(Sx \to Px),\ \forall x(Sx \to \neg Px)\}.$$

This fragment allows us to define $2^{|\mathscr{F}|} = 2^2 = 4$ conjunctions α_i:

$$\begin{array}{rcrcr}
\alpha_1 & := & \forall x(Sx \to Px) & \wedge & \neg\forall x(Sx \to \neg Px) \\
\alpha_2 & := & \neg\forall x(Sx \to Px) & \wedge & \neg\forall x(Sx \to \neg Px) \\
\alpha_3 & := & \neg\forall x(Sx \to Px) & \wedge & \forall x(Sx \to \neg Px) \\
\alpha_4 & := & \forall x(Sx \to Px) & \wedge & \forall x(Sx \to \neg Px)
\end{array}$$

All these conjunctions are FOL-consistent, and hence the partition induced by $\mathscr{F}$ in FOL is $\Pi_{\mathsf{FOL}}(\mathscr{F}) := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Moving from FOL to SYL, however, we notice that α_4 is SYL-inconsistent, and thus we obtain $\Pi_{\mathsf{SYL}}(\mathscr{F}) := \{\alpha_1, \alpha_2, \alpha_3\}$. It should

[35] Observations such as these have led Seuren (2014, p. 505–506) to call FOL an "impoverished system", since some "logical (meta)relations are lost". However, even though the relations may be lost in the concrete Aristotelian square shown in Figure 4(b), they are not lost 'in general', in the sense that FOL still has other formulas standing in those relations. For example, the formulas $\forall x Sx$ and $\forall x \neg Sx$ are FOL-contrary, but happen to be absent from the square in Figure 4(b).

[36] The same situation also arises in modal logic, where systems containing the D-axiom $\Diamond\top$ (which is the modal counterpart of the existential import axiom $\exists x Sx$) yield a classical square, but the minimal normal system K merely yields an 'X of opposition' (Chellas, 1980).

[37] Note that although the I- and O-statement are not in $\mathscr{F}$, they are the Boolean negations of the E- and A-statements, respectively, and thus they *do* belong to the Boolean closure $\mathbb{B}(\mathscr{F})$. The bitstring approach developed in Section 3 will thus allow us to assign bitstrings not only to the A- and E-statements, but also to the I- and O-statements (cf. Theorem 1).

be emphasized that the single fragment $\mathscr{F}$ thus induces two distinct partitions in the logical systems FOL and SYL, which contain resp. 4 and 3 anchor formulas.

We now turn to the bitstring mappings $\beta^{\mathscr{F}}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}$ that are based on the partitions $\Pi_{\mathsf{FOL}}(\mathscr{F})$ and $\Pi_{\mathsf{SYL}}(\mathscr{F})$, respectively. Because these partitions have resp. 4 and 3 anchor formulas, the corresponding bitstring mappings work with bitstrings of length 4 and 3, respectively.[38] For example, for the I-formula $\exists x(Sx \wedge Px)$ we find the following:

$$\begin{array}{llcrll}
\models_{\mathsf{FOL}} & \alpha_1 & \rightarrow & \exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px))]_1 = 1, \\
\models_{\mathsf{FOL}} & \alpha_2 & \rightarrow & \exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px))]_2 = 1, \\
\models_{\mathsf{FOL}} & \alpha_3 & \rightarrow & \neg\exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px))]_3 = 0, \\
\models_{\mathsf{FOL}} & \alpha_4 & \rightarrow & \neg\exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px))]_4 = 0;
\end{array}$$

$$\begin{array}{llcrll}
\models_{\mathsf{SYL}} & \alpha_1 & \rightarrow & \exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge Px))]_1 = 1, \\
\models_{\mathsf{SYL}} & \alpha_2 & \rightarrow & \exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge Px))]_2 = 1, \\
\models_{\mathsf{SYL}} & \alpha_3 & \rightarrow & \neg\exists x(Sx \wedge Px) & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge Px))]_3 = 0.
\end{array}$$

In sum, we find that $\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px)) = 1100$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge Px)) = 110$. It should be noted that a single formula is thus mapped onto two distinct bitstrings, depending on whether this formula is seen as coming from FOL or from SYL. In total, we obtain for the A-, I-, E- and O-statements:

$$\begin{array}{lcllcl}
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)) & = & 1001, & \text{but} & \beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow Px)) & = & 100, \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge Px)) & = & 1100, & \text{but} & \beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge Px)) & = & 110, \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)) & = & 0011, & \text{but} & \beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow \neg Px)) & = & 001, \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\exists x(Sx \wedge \neg Px)) & = & 0110, & \text{but} & \beta^{\mathscr{F}}_{\mathsf{SYL}}(\exists x(Sx \wedge \neg Px)) & = & 011.
\end{array}$$

We are now in a position to study the Aristotelian relations holding between the A-, I-, E- and O-statements in terms of the Aristotelian relations holding between their bitstring representations. For example, for the A- and E-statements we have:

$$\begin{array}{lclcl}
\beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow Px)) \wedge \beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow \neg Px)) & = & 100 \wedge 001 & = & 000, \\
\beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow Px)) \vee \beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow \neg Px)) & = & 100 \vee 001 & \neq & 111.
\end{array}$$

By Definition 2, this means that the bitstrings $\beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow Px))$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}(\forall x(Sx \rightarrow \neg Px))$ are 3-contrary, and hence, it follows by Theorem 2 that the formulas $\forall x(Sx \rightarrow Px)$ and $\forall x(Sx \rightarrow \neg Px)$ are SYL-contrary. By contrast, in FOL it holds that

$$\begin{array}{lcl}
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)) \wedge \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)) & = & 1001 \wedge 0011 = 0001 \neq 0000, \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)) \vee \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)) & = & 1001 \vee 0011 = 1011 \neq 1111, \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)) \wedge \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)) & = & 1001 \wedge 0011 = 0001 \\
 & \neq & \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)), \\
\beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow Px)) \wedge \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)) & = & 1001 \wedge 0011 = 0001 \\
 & \neq & \beta^{\mathscr{F}}_{\mathsf{FOL}}(\forall x(Sx \rightarrow \neg Px)).
\end{array}$$

[38] That bitstrings of length 3 do not suffice in the case of FOL should not come as a big surprise, since the FOL-‘cross’ in Figure 4(b) contains unconnected formulas, and it is well-known that unconnectedness can only be represented by bitstrings of length at least 4 (recall Footnote 12).

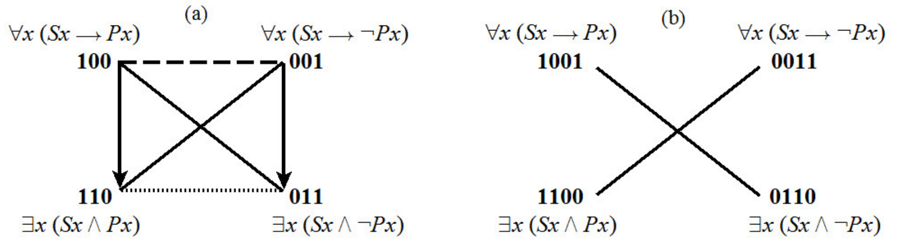

Fig. 5 Aristotelian squares for the categorical statements and their bitstring representations (a) in syllogistics (SYL) and (b) in first-order logic (FOL).

The first item implies that the two formulas are neither FOL-contradictory nor FOL-contrary, and the second one implies that they are neither FOL-contradictory nor FOL-subcontrary. Similarly, the last two items imply that there is no FOL-subalternation from the first formula to the second one, or vice versa. To summarize: in FOL these two formulas do not stand in any Aristotelian relation at all, i.e. they are *unconnected* (recall Footnote 11). It should be noted that a single pair of formulas thus stands in two distinct Aristotelian relations, viz. contrariety in SYL and no Aristotelian relation at all in FOL. Similar remarks apply to all other pairs of A-, I-, E- and O-statements (except for the pairs A/O and I/E, of course, which are SYL-contradictory as well as FOL-contradictory). In sum, then, both the similarities and dissimilarities between the SYL-square and FOL-'cross' from Figure 4 turn out to be systematic consequences of the bitstring mappings $\beta^{\mathscr{F}}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}$, as is illustrated in Figure 5.[39]

From a more general perspective, we see that the addition of the existential import axiom, i.e. the transition from FOL to SYL, changes the anchor formula α_4 from FOL-consistent into SYL-inconsistent. We thus move from $\beta^{\mathscr{F}}_{\mathsf{FOL}}$ to $\beta^{\mathscr{F}}_{\mathsf{SYL}}$, and hence from bitstrings of length 4 to bitstrings of length 3, by systematically deleting the fourth bit position. This deletion process provides a uniform explanation for all the differences between the FOL-cross and SYL-square in Figure 5. First of all, the $\beta^{\mathscr{F}}_{\mathsf{FOL}}$-bitstrings of the A- and E-statements have 1 in their fourth bit position, which is the only position preventing them from being FOL-contrary. In the corresponding $\beta^{\mathscr{F}}_{\mathsf{SYL}}$-bitstrings, however, the fourth bit position is deleted, and the A- and E-statements turn out to be SYL-contrary. Secondly, the I- and O-statements are not FOL-subcontrary only because their $\beta^{\mathscr{F}}_{\mathsf{FOL}}$-bitstrings have 0 in their fourth bit position. The deletion of this position in $\beta^{\mathscr{F}}_{\mathsf{SYL}}$ thus leads to these statements being SYL-subcontrary. Finally, there is no FOL-subalternation from the A- to the I-statement only because their $\beta^{\mathscr{F}}_{\mathsf{FOL}}$-bitstrings have resp. 1 and 0 in their fourth bit position, and hence deleting this position in $\beta^{\mathscr{F}}_{\mathsf{SYL}}$ leads to the traditional SYL-subalternation (similar remarks apply to the E- and O-statements).

It should be noted that the case described in this section is somewhat similar to that described in Example 7. In that example, we considered the fragment $\mathscr{F}^{\dagger}_{**}$ and assigned bitstrings to it using two distinct mappings $\beta^{\mathscr{F}^{\dagger}}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}^{\dagger}_{**}}_{\mathsf{FOL}}$, which work with

[39] In Peirce's logical writings (1932), we already find what essentially amounts to a FOL-based bitstring semantics for the categorical statements in terms of bitstrings of length 4 (CP 2.456), immediately followed by the observation that this does not yield a classical square of opposition, but rather a 'cross' (CP 2.460).

bitstrings of length 4 and 3, respectively. In the present section, too, we considered a single fragment $\mathscr{F}$ and two distinct mappings $\beta^{\mathscr{F}}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}$, which also work with bitstrings of length 4 and 3, respectively. Despite these similarities in bitstring length, there is a fundamental conceptual difference between the two pairs of bitstring mappings. In Example 7, the bitstring mappings $\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}^\dagger_{**}}_{\mathsf{FOL}}$ are induced by two *distinct fragments* (viz. $\mathscr{F}^\dagger_{**}$ and $\mathscr{F}^\dagger$) but within a *single logic* (viz. FOL). In other words, the underlying partitions $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger_{**})$ and $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$ are two partitions of the same logical space, but with different levels of *granularity*: the two anchor formulas $Pa \wedge \neg\forall xPx$ and $\neg Pa \wedge \exists xPx$ of the quadripartition $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger)$ are *collapsed* into a single new anchor formula in the tripartition $\Pi_{\mathsf{FOL}}(\mathscr{F}^\dagger_{**})$, viz. $\exists xPx \wedge \exists x\neg Px$ (recall Footnote 28). By contrast, in the case studied in the present section, the bitstring mappings $\beta^{\mathscr{F}}_{\mathsf{FOL}}$ and $\beta^{\mathscr{F}}_{\mathsf{SYL}}$ are induced by a *single fragment* (viz. $\mathscr{F}$) but operate within two *different logics* (viz. FOL and SYL). In other words, the underlying partitions $\Pi_{\mathsf{FOL}}(\mathscr{F})$ and $\Pi_{\mathsf{SYL}}(\mathscr{F})$ are partitions of two different logical spaces: the anchor formula α_4 of the quadripartition $\Pi_{\mathsf{FOL}}(\mathscr{F})$ is *deleted* to yield the tripartition $\Pi_{\mathsf{SYL}}(\mathscr{F})$.[40]

The fact that the logical systems SYL and FOL represent the fragment $\mathscr{F}$ with bitstrings of different lengths, also means that this fragment has distinct Boolean closures in these two logical systems. This is a direct consequence of the fact that the squares in Figure 4 are not Aristotelian isomorphic, and hence, by Lemma 1, not Boolean isomorphic either. In particular, in SYL the Boolean closure of $\mathscr{F}$ is a Boolean algebra $\mathbb{B}_3$ containing $2^3 = 8$ formulas, whereas in FOL it is a Boolean algebra $\mathbb{B}_4$ containing $2^4 = 16$ formulas. In terms of Aristotelian diagrams—i.e. ignoring the contradictory and tautologous propositions (recall Footnote 14)—, the Boolean closure of the SYL-square in Figure 5(a) is a JSB hexagon, whereas that of the FOL-cross in Figure 5(b) is a rhombic dodecahedron.

Finally, it should be noted that the bitstring approach developed in Section 3 turns out to be intricately related to Boolean closures. On the one hand, we do not have to determine the fragment $\mathscr{F}$'s Boolean closure in order to define a bitstring semantics $\beta^{\mathscr{F}}_{\mathsf{S}}$ for it (where S can be either FOL or SYL); on the other hand, once defined, the bitstring semantics $\beta^{\mathscr{F}}_{\mathsf{S}}$ does enable us to study the Boolean closure of $\mathscr{F}$ and its properties (e.g. its size).

5 Aristotelian and Boolean Structure

5.1 Strong and Weak Jacoby-Sesmat-Blanché Hexagons

We will now return to the second problem that was raised in Subsection 2.3, which concerned the exact nature of the interplay between Boolean and Aristotelian structure. It was already shown in Subsection 2.1 that Aristotelian structure is determined

[40] The difference between the two pairs of bitstring mappings is also manifest in their typographic rendering: in Example 7 we have different values for the fragment parameter in *superscript* ($\beta^{\mathscr{F}^\dagger}_{\mathsf{FOL}}$ vs. $\beta^{\mathscr{F}^\dagger_{**}}_{\mathsf{FOL}}$), whereas in the present section we have different values for the logic parameter in *subscript* ($\beta^{\mathscr{F}}_{\mathsf{FOL}}$ vs. $\beta^{\mathscr{F}}_{\mathsf{SYL}}$).

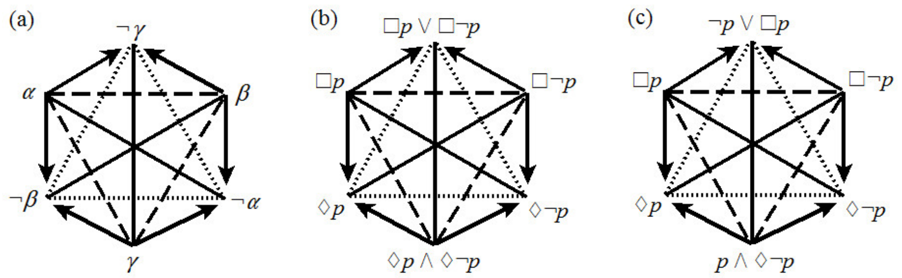

Fig. 6 (a) General format of a JSB hexagon, (b) example of a strong JSB hexagon, (c) example of a weak JSB hexagon.

by Boolean structure, and thus, that every Boolean isomorphism is also an Aristotelian isomorphism (cf. Lemma 1). The converse, however, does not hold: there exist pairs of diagrams that are Aristotelian isomorphic to each other, but not Boolean isomorphic. In this subsection and the next one, we will discuss two such diagram pairs.

The first example concerns the so-called *strong* and *weak* variants of the Jacoby-Sesmat-Blanché (JSB) hexagon that was introduced in Subsection 2.2. In general, a JSB hexagon consists of a contrariety triangle (α–β–γ) interlocked with a subcontrariety triangle ($\neg\alpha$–$\neg\beta$–$\neg\gamma$), as is shown in Figure 6(a). Pellissier (2008, p. 238–239) defines a JSB hexagon to be *strong* iff the disjunction of the formulas on its contrariety triangle is an S-tautology, i.e. $\models_{\mathsf{S}} \alpha \vee \beta \vee \gamma$; likewise, a JSB hexagon is said to be *weak* iff it is not strong.[41] For example, the JSB hexagon with S5-formulas in Figure 6(b) is *strong*, since $\models_{\mathsf{S5}} \Box p \vee \Box\neg p \vee (\Diamond p \wedge \Diamond\neg p)$, whereas the JSB hexagon in Figure 6(c) is *weak*, since $\not\models_{\mathsf{S5}} \Box p \vee \Box\neg p \vee (p \wedge \Diamond\neg p)$. Letting $\mathscr{F}_1$ and $\mathscr{F}_2$ be the sets of formulas appearing in the JSB hexagons in Figures 6(b) and (c), respectively, we see that these two JSB hexagons are *Aristotelian isomorphic*: one can check that the function $\rho\colon \mathscr{F}_1 \to \mathscr{F}_2$ defined as

$\varphi \in \mathscr{F}_1$	$\Box p$	$\Box\neg p$	$\Diamond p$	$\Diamond\neg p$	$\Diamond p \wedge \Diamond\neg p$	$\Box p \vee \Box\neg p$
$\rho(\varphi) \in \mathscr{F}_2$	$\Box p$	$\Box\neg p$	$\Diamond p$	$\Diamond\neg p$	$p \wedge \Diamond\neg p$	$\neg p \vee \Box p$

is an Aristotelian isomorphism (recall Definition 4). However, these two JSB hexagons are not *Boolean isomorphic*, since there is no Boolean isomorphism $\iota\colon \mathscr{F}_1 \to \mathscr{F}_2$. For example, the mapping ρ defined above is not a Boolean isomorphism, since for $\Box p, \Box\neg p$ and $\Box p \vee \Box\neg p$ in $\mathscr{F}_1$ it trivially holds that the latter is S5-equivalent to the disjunction of the former two, whereas for their ρ-images in $\mathscr{F}_2$ we have $\rho(\Box p) \vee \rho(\Box\neg p) = \Box p \vee \Box\neg p \not\equiv_{\mathsf{S5}} \neg p \vee \Box p = \rho(\Box p \vee \Box\neg p)$.

Applying the bitstring approach described in Section 3, we find that the fragment $\mathscr{F}_1$ induces the tripartition $\Pi_{\mathsf{S5}}(\mathscr{F}_1) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$, while the fragment $\mathscr{F}_2$ induces the quadripartition $\Pi_{\mathsf{S5}}(\mathscr{F}_2) = \{\Box p, p \wedge \Diamond\neg p, \neg p \wedge \Diamond p, \Box\neg p\}$. The bitstring mappings based on these partitions are $\beta^{\mathscr{F}_1}_{\mathsf{S5}}\colon \mathbb{B}(\mathscr{F}_1) \to \{0,1\}^3$ and $\beta^{\mathscr{F}_2}_{\mathsf{S5}}\colon \mathbb{B}(\mathscr{F}_2) \to \{0,1\}^4$. We can now calculate the bitstrings $\beta^{\mathscr{F}_1}_{\mathsf{S5}}(\varphi)$ and $\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\psi)$

[41] Equivalently, one can also define a JSB hexagon to be strong iff the conjunction of the formulas on its subcontrariety triangle is an S-contradiction, i.e. $\models_{\mathsf{S}} \neg(\neg\alpha \wedge \neg\beta \wedge \neg\gamma)$.

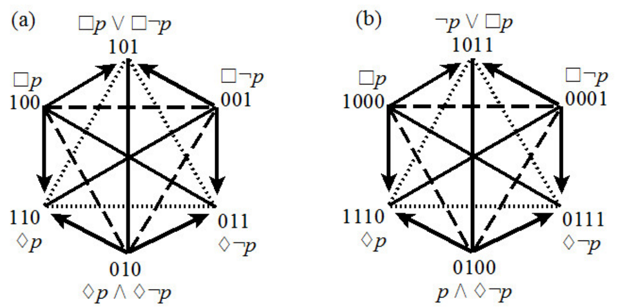

Fig. 7 The two JSB hexagons and their bitstrings representations.

for all $\varphi \in \mathscr{F}_1$ and $\psi \in \mathscr{F}_2$. Furthermore, since $\mathscr{F}_1 \subseteq \mathbb{B}(\mathscr{F}_2)$, we can also calculate the bitstrings $\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\varphi)$ for all $\varphi \in \mathscr{F}_1$.[42] The resulting bitstrings are listed in the table below, and used in Figure 7 to decorate the strong and weak JSB hexagons.

$\varphi \in \mathscr{F}_1$	$\beta^{\mathscr{F}_1}_{\mathsf{S5}}(\varphi)$	$\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\varphi)$	$\psi \in \mathscr{F}_2$	$\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\psi)$
$\Box p$	100	1000	$\Box p$	1000
$\Box\neg p$	001	0001	$\Box\neg p$	0001
$\Diamond p$	110	1110	$\Diamond p$	1110
$\Diamond\neg p$	011	0111	$\Diamond\neg p$	0111
$\Diamond p \wedge \Diamond\neg p$	010	0110	$p \wedge \Diamond\neg p$	0100
$\Box\neg p \vee \Box p$	101	1001	$\neg p \vee \Box p$	1011

It should be noted that the formulas of $\mathscr{F}_1$ constitute a *strong* JSB hexagon, regardless of whether they are encoded in length 3 by $\beta^{\mathscr{F}_1}_{\mathsf{S5}}$ or in length 4 by $\beta^{\mathscr{F}_2}_{\mathsf{S5}}$: in the former case we have $\beta^{\mathscr{F}_1}_{\mathsf{S5}}(\Box p) \vee \beta^{\mathscr{F}_1}_{\mathsf{S5}}(\Box\neg p) \vee \beta^{\mathscr{F}_1}_{\mathsf{S5}}(\Diamond p \wedge \Diamond\neg p) = 100 \vee 001 \vee 010 = 111$, and in the latter $\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\Box p) \vee \beta^{\mathscr{F}_2}_{\mathsf{S5}}(\Box\neg p) \vee \beta^{\mathscr{F}_2}_{\mathsf{S5}}(\Diamond p \wedge \Diamond\neg p) = 1000 \vee 0001 \vee 0110 = 1111$. By contrast, the formulas of $\mathscr{F}_2$ constitute a *weak* JSB hexagon: $\beta^{\mathscr{F}_2}_{\mathsf{S5}}(\Box p) \vee \beta^{\mathscr{F}_2}_{\mathsf{S5}}(\Box\neg p) \vee \beta^{\mathscr{F}_2}_{\mathsf{S5}}(p \wedge \Diamond\neg p) = 1000 \vee 0001 \vee 0100 = 1101 \neq 1111$.

It was shown in Subsection 2.1 that every Boolean isomorphism is an Aristotelian isomorphism. Clearly, the converse does not hold: there is an Aristotelian isomorphism between the fragments $\mathscr{F}_1$ and $\mathscr{F}_2$, but not a Boolean isomorphism. An easy way to see that $\mathscr{F}_1$ and $\mathscr{F}_2$ are not Boolean isomorphic is by noting that the minimal bitstring semantics of these fragments make use of different lengths, viz. 3 and 4, and hence, they have different Boolean closures, which is in direct contradiction to Definition 4 (recall Footnote 10).

The Boolean closure of a strong JSB hexagon is $\mathbb{B}_3$, which contains $2^3 - 2 = 6$ contingent formulas (recall Footnote 14). The Boolean closure of a strong JSB hexagon is thus that hexagon itself, which means exactly that the strong JSB hexagon is Boolean closed (†). On the other hand, the Boolean closure of a weak JSB hexagon is $\mathbb{B}_4$, which contains $2^4 - 2 = 14$ contingent formulas. The Boolean closure of a weak

[42] Since $p \wedge \Diamond\neg p \in \mathscr{F}_2$ but $p \wedge \Diamond\neg p \notin \mathbb{B}(\mathscr{F}_1)$, it holds that $\mathscr{F}_2 \not\subseteq \mathbb{B}(\mathscr{F}_1)$, and we thus cannot define bitstrings $\beta^{\mathscr{F}_1}_{\mathsf{S5}}(\varphi)$ for $\varphi \in \mathscr{F}_2$. Finally, note the analogy between $\mathscr{F}_1$ and $\mathscr{F}_2$ here and $\mathscr{F}^{\dagger}_{**}$ and $\mathscr{F}^{\dagger}$ in Example 7.

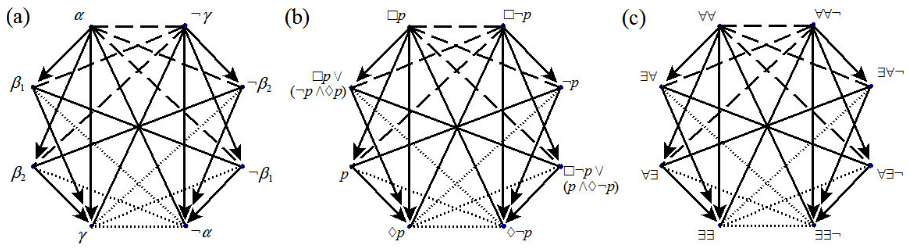

Fig. 8 (a) General format of a Buridan octagon, and examples of Buridan octagons with formulas from (b) S5 and (c) FOL.

JSB hexagon is thus a different diagram (viz. an RDH) than that hexagon itself, and hence the weak JSB hexagon is *not* Boolean closed. By contraposition, this means that if a JSB hexagon is Boolean closed, then it is not weak, i.e. strong (‡). Putting (†) and (‡) together, we see that a JSB hexagon is strong iff it is Boolean closed. Although this property of JSB hexagons was already known (Smessaert and Demey, 2017), the bitstring approach presented in this paper allows us to prove it in a simpler and conceptually more elegant manner.

Finally, the interplay between Aristotelian and Boolean structure is also manifested in the terminology used to label Aristotelian diagrams. On the one hand, the strong and weak JSB hexagons can be characterized as *subtypes of a single Aristotelian family*,[43] viz. the JSB family: although a strong and a weak JSB hexagon are not Boolean isomorphic, they are Aristotelian isomorphic to each other. On the other hand, the (strong/weak) JSB hexagons, the SC hexagons and the U4 hexagons (recall Footnote 13) are *three distinct Aristotelian families*: a JSB hexagon, an SC hexagon and a U4 hexagon are pairwise neither Boolean nor Aristotelian isomorphic to each other. In other words, the difference between a strong and a weak JSB hexagon is less substantial than the pairwise differences between a JSB, an SC and a U4 hexagon.

5.2 Types of Buridan Octagons

We now turn to the second example of fragments/diagrams that are Aristotelian isomorphic to each other, but not Boolean isomorphic. A Buridan octagon is an Aristotelian diagram containing 8 formulas, with a general configuration as shown in Figure 8(a) (recall Footnote 15). Various logical systems give rise to this type of configuration; for example, Figure 8(b) shows a Buridan octagon with formulas from S5, while Figure 8(c) shows a Buridan octagon with FOL-formulas of the form $Q_1xQ_2yR(x,y)$ and $Q_1xQ_2y\neg R(x,y)$, with $Q_1, Q_2 \in \{\forall, \exists\}$.[44] Letting $\mathscr{F}_1$ and $\mathscr{F}_2$ be the sets of formulas appearing in Figures 8(b) and 8(c), respectively, we see that these

[43] Two diagrams are said to belong to the same *Aristotelian family* iff they are Aristotelian isomorphic to each other.

[44] In this section, we will write Q_1Q_2 for the formula $Q_1xQ_2yR(x,y)$ and $Q_1Q_2\neg$ for the formula $Q_1xQ_2y\neg R(x,y)$, with $Q_1, Q_2 \in \{\forall, \exists\}$. For example, $\forall x\exists yR(x,y)$ and $\exists x\forall y\neg R(x,y)$ will be abbreviated as $\forall\exists$ and $\exists\forall\neg$, respectively. Finally, note that John Buridan himself already made use of formulas involving multiple quantifiers to construct a Buridan octagon. For example, he considered sentences

two Buridan octagons are *Aristotelian isomorphic*: one can check that the function $\rho\colon \mathscr{F}_1 \to \mathscr{F}_2$ defined by

$\varphi \in \mathscr{F}_1$	$\Box p$	$\Box\neg p$	$\Diamond p$	$\Diamond\neg p$	p	$\Box\neg p \vee (p \wedge \Diamond\neg p)$	$\Box p \vee (\neg p \wedge \Diamond p)$	$\neg p$
$\rho(\varphi) \in \mathscr{F}_2$	$\forall\forall$	$\forall\forall\neg$	$\exists\exists$	$\exists\exists\neg$	$\forall\exists$	$\forall\exists\neg$	$\exists\forall$	$\exists\forall\neg$

is an Aristotelian isomorphism (recall Definition 4). However, these Buridan octagons are not *Boolean isomorphic*, since there is no Boolean isomorphism $\iota\colon \mathscr{F}_1 \to \mathscr{F}_2$. For example, the mapping ρ defined above is not a Boolean isomorphism, since for p and $\Box p \vee (\neg p \vee \Diamond p)$ in $\mathscr{F}_1$ it holds that $p \wedge (\Box p \vee (\neg p \wedge \Diamond p)) \equiv_{\mathsf{S5}} \Box p$ and also that $p \vee (\Box p \vee (\neg p \wedge \Diamond p)) \equiv_{\mathsf{S5}} \Diamond p$, but for their ρ-images in $\mathscr{F}_2$ we have $\rho(p) \wedge \rho(\Box p \vee (\neg p \wedge \Diamond p)) = \forall\exists \wedge \exists\forall \not\equiv_{\mathsf{FOL}} \forall\forall = \rho(\Box p)$ and also $\rho(p) \vee \rho(\Box p \vee (\neg p \wedge \Diamond p)) = \forall\exists \vee \exists\forall \not\equiv_{\mathsf{FOL}} \exists\exists = \rho(\Diamond p)$.

We will now apply the bitstring approach described in Section 3 to these Buridan octagons. However, instead of studying the partitions induced by the fragments $\mathscr{F}_1$ and $\mathscr{F}_2$ directly, we will work in a more 'incremental' fashion. We start by defining the subfragments $\mathscr{F}_1^a := \{\Box p, \Box\neg p, \Diamond p, \Diamond\neg p\}$ and $\mathscr{F}_1^b := \{p, \Box\neg p \vee (p \wedge \Diamond\neg p), \Box p \vee (\neg p \wedge \Diamond p), \neg p\}$ of $\mathscr{F}_1$, which correspond to the 'vertically stretched' square of opposition and the 'horizontally stretched' X of opposition inside the Buridan octagon in Figure 8(b), respectively. These subfragments induce a tripartition and a quadripartition:

- $\Pi_{\mathsf{S5}}(\mathscr{F}_1^a) = \{\Box p, \Diamond p \wedge \Diamond\neg p, \Box\neg p\}$,
- $\Pi_{\mathsf{S5}}(\mathscr{F}_1^b) = \{\Box p, p \wedge \Diamond\neg p, \neg p \wedge \Diamond p, \Box\neg p\}$.

Completely analogously, we also define the subfragments $\mathscr{F}_2^a := \rho[\mathscr{F}_1^a] = \{\forall\forall, \forall\forall\neg, \exists\exists, \exists\exists\neg\}$ and $\mathscr{F}_2^b := \rho[\mathscr{F}_1^b] = \{\forall\exists, \forall\exists\neg, \exists\forall, \exists\forall\neg\}$ of $\mathscr{F}_2$, which correspond to the vertically and horizontally stretched square and X of oppositions inside the octagon in Figure 8(c), respectively. These subfragments also induce a tripartition and a quadripartition:

- $\Pi_{\mathsf{FOL}}(\mathscr{F}_2^a) = \{\forall\forall, \exists\exists \wedge \exists\exists\neg, \forall\forall\neg\}$
- $\Pi_{\mathsf{FOL}}(\mathscr{F}_2^b) = \{\forall\exists \wedge \exists\forall, \forall\exists \wedge \forall\exists\neg, \exists\forall \wedge \exists\forall\neg, \forall\exists\neg \wedge \exists\forall\neg\}$.

Since $\mathscr{F}_1^a \cup \mathscr{F}_1^b = \mathscr{F}_1$ and $\mathscr{F}_2^a \cup \mathscr{F}_2^b = \mathscr{F}_2$, it follows by Lemma 4 that $\Pi_{\mathsf{S5}}(\mathscr{F}_1) = \Pi_{\mathsf{S5}}(\mathscr{F}_1^a) \wedge_{\mathsf{S5}} \Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$ and $\Pi_{\mathsf{FOL}}(\mathscr{F}_2) = \Pi_{\mathsf{FOL}}(\mathscr{F}_2^a) \wedge_{\mathsf{FOL}} \Pi_{\mathsf{FOL}}(\mathscr{F}_2^b)$. Following Definition 6, we start by putting each formula of $\Pi_{\mathsf{S5}}(\mathscr{F}_1^a)$ in conjunction with each formula of $\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$, and similarly for $\Pi_{\mathsf{FOL}}(\mathscr{F}_2^a)$ and $\Pi_{\mathsf{FOL}}(\mathscr{F}_2^b)$. As to $\Pi_{\mathsf{S5}}(\mathscr{F}_1)$, we see that 8 of the 12 conjunctions are $\mathsf{S5}$-inconsistent, viz. those whose row number has a $\perp$-superscript in the left table below. By contrast, for $\Pi_{\mathsf{FOL}}(\mathscr{F}_2)$, we see that only 6 of the 12 conjunctions are FOL-inconsistent, viz. those whose row number has a $\perp$-superscript in the right table below.[45]

of the form "of every human, every donkey runs" (Read, 2012, p. 107), which can be formalized as $\forall x\Big(human(x) \to \forall y\big((donkey(y) \wedge own(x,y)) \to run(y)\big)\Big)$.

[45] Note, in particular, that the conjunctions on rows 5 and 8 are FOL-consistent.

	$\Pi_{\mathsf{S5}}(\mathscr{F}_1^a)$	$\wedge_{\mathsf{S5}}$	$\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$	$\Pi_{\mathsf{FOL}}(\mathscr{F}_2^a)$	$\wedge_{\mathsf{FOL}}$	$\Pi_{\mathsf{FOL}}(\mathscr{F}_2^b)$	
1	$\Box p$	$\wedge$	$\Box p$	$\forall\forall$	$\wedge$	$\forall\exists\wedge\exists\forall$	1
$2^{\perp}$	$\Box p$	$\wedge$	$p\wedge\Diamond\neg p$	$\forall\forall$	$\wedge$	$\forall\exists\wedge\forall\exists\neg$	$2^{\perp}$
$3^{\perp}$	$\Box p$	$\wedge$	$\neg p\wedge\Diamond p$	$\forall\forall$	$\wedge$	$\exists\forall\wedge\exists\forall\neg$	$3^{\perp}$
$4^{\perp}$	$\Box p$	$\wedge$	$\Box\neg p$	$\forall\forall$	$\wedge$	$\forall\exists\neg\wedge\exists\forall\neg$	$4^{\perp}$
$5^{\perp}$	$\Diamond p\wedge\Diamond\neg p$	$\wedge$	$\Box p$	$\exists\exists\wedge\exists\exists\neg$	$\wedge$	$\forall\exists\wedge\exists\forall$	5
6	$\Diamond p\wedge\Diamond\neg p$	$\wedge$	$p\wedge\Diamond\neg p$	$\exists\exists\wedge\exists\exists\neg$	$\wedge$	$\forall\exists\wedge\forall\exists\neg$	6
7	$\Diamond p\wedge\Diamond\neg p$	$\wedge$	$\neg p\wedge\Diamond p$	$\exists\exists\wedge\exists\exists\neg$	$\wedge$	$\exists\forall\wedge\exists\forall\neg$	7
$8^{\perp}$	$\Diamond p\wedge\Diamond\neg p$	$\wedge$	$\Box\neg p$	$\exists\exists\wedge\exists\exists\neg$	$\wedge$	$\forall\exists\neg\wedge\exists\forall\neg$	8
$9^{\perp}$	$\Box\neg p$	$\wedge$	$\Box p$	$\forall\forall\neg$	$\wedge$	$\forall\exists\wedge\exists\forall$	$9^{\perp}$
$10^{\perp}$	$\Box\neg p$	$\wedge$	$p\wedge\Diamond\neg p$	$\forall\forall\neg$	$\wedge$	$\forall\exists\wedge\forall\exists\neg$	$10^{\perp}$
$11^{\perp}$	$\Box\neg p$	$\wedge$	$\neg p\wedge\Diamond p$	$\forall\forall\neg$	$\wedge$	$\exists\forall\wedge\exists\forall\neg$	$11^{\perp}$
12	$\Box\neg p$	$\wedge$	$\Box\neg p$	$\forall\forall\neg$	$\wedge$	$\forall\exists\neg\wedge\exists\forall\neg$	12

After simplifying the remaining formulas, we find:[46]

- $\Pi_{\mathsf{S5}}(\mathscr{F}_1)=\{\Box p, p\wedge\Diamond\neg p, \neg p\wedge\Diamond p, \Box\neg p\}$,
- $\Pi_{\mathsf{S5}}(\mathscr{F}_2)=\{\forall\forall, \forall\exists\wedge\exists\forall\wedge\exists\exists\neg, \forall\exists\wedge\forall\exists\neg, \exists\forall\wedge\exists\forall\neg, \forall\exists\neg\wedge\exists\forall\neg\wedge\exists\exists, \forall\forall\neg\}$.

The bitstring mappings based on these partitions are $\beta_{\mathsf{S5}}^{\mathscr{F}_1}:\mathbb{B}(\mathscr{F}_1)\to\{0,1\}^4$ and $\beta_{\mathsf{FOL}}^{\mathscr{F}_2}:\mathbb{B}(\mathscr{F}_2)\to\{0,1\}^6$. We can now calculate the bitstrings $\beta_{\mathsf{S5}}^{\mathscr{F}_1}(\varphi)$ and $\beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\psi)$ for all $\varphi\in\mathscr{F}_1$ and $\psi\in\mathscr{F}_2$. The resulting bitstrings are listed in the table below, and used in Figure 9 to decorate the two Buridan octagons.

$\varphi\in\mathscr{F}_1$	$\beta_{\mathsf{S5}}^{\mathscr{F}_1}(\varphi)$	$\psi\in\mathscr{F}_2$	$\beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\psi)$
$\Box p$	1000	$\forall\forall$	100000
$\Box\neg p$	0001	$\forall\forall\neg$	000001
$\Diamond p$	1110	$\exists\exists$	111110
$\Diamond\neg p$	0111	$\exists\exists\neg$	011111
p	1100	$\forall\exists$	111000
$\Box\neg p\vee(p\wedge\Diamond\neg p)$	0101	$\forall\exists\neg$	001011
$\Box p\vee(\neg p\wedge\Diamond p)$	1010	$\exists\forall$	110100
$\neg p$	0011	$\exists\forall\neg$	000111

The fact that the minimal bitstring semantics of $\mathscr{F}_1$ and $\mathscr{F}_2$ make use of different lengths, viz. 4 and 6, means that these fragments have different Boolean closures, and hence are not Boolean isomorphic. This was already illustrated above by means of the formulas $p, \Box p\vee(\neg p\wedge\Diamond p), \Box p, \Diamond p\in\mathscr{F}_1$ and their ρ-images $\forall\exists, \exists\forall, \forall\forall, \exists\exists\in\mathscr{F}_2$; in terms of bitstrings, we have:

$$\begin{array}{lclclclclcl}
\beta_{\mathsf{S5}}^{\mathscr{F}_1}(p) & \wedge & \beta_{\mathsf{S5}}^{\mathscr{F}_1}(\Box p\vee(\neg p\wedge\Diamond p)) & = & 1100 & \wedge & 1010 & = & 1000 & = & \beta_{\mathsf{S5}}^{\mathscr{F}_1}(\Box p), \\
\beta_{\mathsf{S5}}^{\mathscr{F}_1}(p) & \vee & \beta_{\mathsf{S5}}^{\mathscr{F}_1}(\Box p\vee(\neg p\wedge\Diamond p)) & = & 1100 & \vee & 1010 & = & 1110 & = & \beta_{\mathsf{S5}}^{\mathscr{F}_1}(\Diamond p)
\end{array}$$

$$\begin{array}{lclclclclcl}
\beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\forall\exists) & \wedge & \beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\exists\forall) & = & 111000 \wedge 110100 & = & 110000 & \neq & 100000 & = & \beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\forall\forall), \\
\beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\forall\exists) & \vee & \beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\exists\forall) & = & 111000 \vee 110100 & = & 111100 & \neq & 111110 & = & \beta_{\mathsf{FOL}}^{\mathscr{F}_2}(\exists\exists).
\end{array}$$

[46] Note that $\Pi_{\mathsf{S5}}(\mathscr{F}_1)=\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$; the reason for this is that although $\mathscr{F}_1^a\not\subseteq\mathscr{F}_1^b$, it does hold that $\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$ is a *refinement* of $\Pi_{\mathsf{S5}}(\mathscr{F}_1^a)$ (in the sense of Lemma 5), and hence $\Pi_{\mathsf{S5}}(\mathscr{F}_1)=\Pi_{\mathsf{S5}}(\mathscr{F}_1^a)\wedge_{\mathsf{S5}}\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)=\Pi_{\mathsf{S5}}(\mathscr{F}_1^b)$ (also recall Footnote 22).

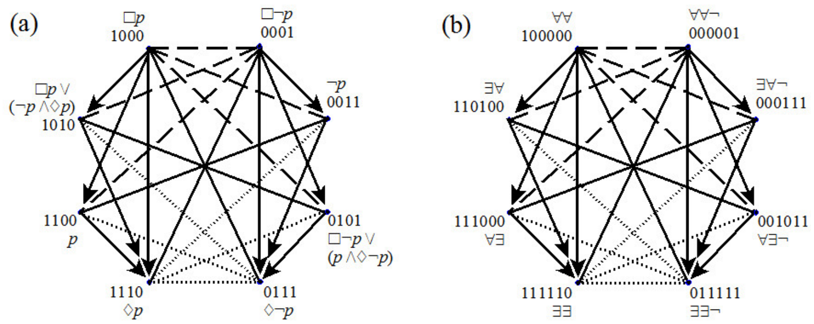

Fig. 9 The two Buridan octagons and their bitstrings representations.

We see that in terms of bitstrings, the facts that $\forall\exists \wedge \exists\forall \not\equiv_{\mathsf{FOL}} \forall\forall$ and $\forall\exists \vee \exists\forall \not\equiv_{\mathsf{FOL}} \exists\exists$ are due to the second and fifth bit positions, respectively, which correspond to the anchor formulas $\forall\exists \wedge \exists\forall \wedge \exists\exists\neg$ and $\forall\exists\neg \wedge \exists\forall\neg \wedge \exists\exists$ in $\Pi_{\mathsf{FOL}}(\mathscr{F}_2)$. Note that these are exactly (equivalent to) the conjunctions on rows 5 and 8 of the table given above, i.e. the conjunctions of $\mathscr{F}_2$-formulas that are FOL-consistent, but whose $\mathscr{F}_1$-counterparts are $\mathsf{S5}$-inconsistent (recall Footnote 45). The Boolean differences between $\mathscr{F}_1$ and $\mathscr{F}_2$ are thus directly reflected in the lengths of their minimal bitstring representations.

Moving to a more abstract level, and using the formula labels $\alpha, \beta_1, \beta_2, \gamma$ as in Figure 8(a), we can distinguish the following four cases:

1. $\alpha \equiv \beta_1 \wedge \beta_2$ and $\gamma \equiv \beta_1 \vee \beta_2$ (requiring bitstrings of length 4),
2. $\alpha \equiv \beta_1 \wedge \beta_2$ and $\gamma \not\equiv \beta_1 \vee \beta_2$ (requiring bitstrings of length 5),
3. $\alpha \not\equiv \beta_1 \wedge \beta_2$ and $\gamma \equiv \beta_1 \vee \beta_2$ (requiring bitstrings of length 5),
4. $\alpha \not\equiv \beta_1 \wedge \beta_2$ and $\gamma \not\equiv \beta_1 \vee \beta_2$ (requiring bitstrings of length 6).

As was already discussed above, cases 1 and 4 are exemplified by the Buridan octagons in Figure 9(a) and (b), respectively. Furthermore, one can show that the Buridan octagons defined in cases 2 and 3 are Boolean isomorphic to each other.[47] Since the bitstring representations of the Buridan octagons in cases 1, 2/3 and 4 require different bitstring lengths, it follows immediately that these Buridan octagons are not Boolean isomorphic to each other. Summing up: the Aristotelian family of Buridan octagons comes in three subtypes, which are pairwise Aristotelian, but not Boolean isomorphic to each other. This insight is not entirely new, but the bitstring approach developed in this paper leads to a more comprehensive perspective, and allows us to systematically relate these subtypes to the different bitstring lengths that they give rise to.[48]

[47] Assume that the Buridan octagons in cases 2 and 3 have the formulas $\{\alpha, \beta_1, \beta_2, \gamma\}$ and $\{\alpha', \beta_1', \beta_2', \gamma'\}$ as their respective left-hand-sides (see Figure 8(a)). A concrete Boolean isomorphism ι between these Buridan octagons looks as follows: $\iota(\alpha) = \neg\gamma'$, $\iota(\beta_i) = \neg\beta_i'$ for $i \in \{1,2\}$, $\iota(\gamma) = \neg\alpha'$, and $\iota(\neg\varphi) = \neg\iota(\varphi')$ for $\varphi \in \{\alpha, \beta_1, \beta_2, \gamma\}$. Informally, ι thus maps the left-hand-side of the first Buridan octagon onto the right-hand-side of the second one, and vice versa.

[48] The first subtype, in which $\alpha \equiv \beta_1 \wedge \beta_2$ and $\gamma \equiv \beta_1 \vee \beta_2$, has also been called a *rhombicube* (Smessaert and Demey, 2014a, 2015b,c), because it can be represented by bitstrings of length 4, and can thus be embedded inside the Aristotelian rhombic dodecahedron (which visualizes $\mathbb{B}_4$).

Finally, we would like to emphasize that the ‘incremental’ methodology used in this subsection is not only useful from a purely practical perspective, but can also help to precisely locate the Boolean differences between Aristotelian isomorphic diagrams. The Buridan octagons in Figure 8(b) and (c) can both be decomposed into a ‘vertically stretched’ square and a ‘horizontally stretched’ X of oppositions, and in both cases, the former induces a tripartition and the latter a quadripartition. Partitional differences become only manifest when the meets of the tri- and quadripartitions are calculated. This shows that the Boolean differences between the various subtypes of Buridan octagons do not arise *locally* within the octagon’s vertically and horizontally stretched square components, but only in the *global interaction* between those components.

6 Bitstring Semantics for Public Announcement Logic

We will now address the third and final problem that was mentioned in Subsection 2.3, viz. the issue of systematicity. Although the original bitstring approach has been applied successfully to study several logical systems and fragments, it could not straightforwardly be generalized to new logical systems and/or fragments. By contrast, the bitstring approach presented in this paper is fully general, which we will now illustrate by showing how it allows us to analyze a fragment from the system of public announcement logic.

Public announcement logic (PAL) belongs to the broader family of dynamic epistemic logics, which allow us to reason about agents’ knowledge and how it is influenced by epistemically relevant actions (Plaza, 1989; Gerbrandy and Groeneveld, 1997; van Ditmarsch et al, 2007). For example, a public announcement of a proposition p can cause agents to *learn* (i.e. to gain knowledge): even if an agent did not know that p before the public announcement, she does know that p after the announcement. In PAL, announcements are always assumed to be *truthful*, i.e. only true propositions can be announced. The language of PAL contains formulas of the form $[!\varphi]\psi$, which informally means that if φ can be announced at all (i.e. if φ is true), then ψ will be the case after this announcement. The dual statement is $\langle !\varphi\rangle\psi := \neg[!\varphi]\neg\psi$, which informally means that φ can indeed be announced (i.e. φ is indeed true), and ψ will be the case after this announcement. This is formalized in the semantics of PAL, which makes use of (updates on) Kripke models:

$$\begin{array}{lll} \mathbb{M},w \models [!\varphi]\psi & \text{iff} & \text{if } \mathbb{M},w \models \varphi \text{ then } \mathbb{M}|\varphi,w \models \psi, \\ \mathbb{M},w \models \langle !\varphi\rangle\psi & \text{iff} & \mathbb{M},w \models \varphi \text{ and } \mathbb{M}|\varphi,w \models \psi. \end{array}$$

Note that in these clauses, the formula ψ is not interpreted on the original model $\mathbb{M}$ itself (which represents the agents’ knowledge *before* the announcement of φ), but rather on the updated model $\mathbb{M}|\varphi$ (which represents the agents’ knowledge *after* the announcement of φ). The precise definition of this model update operation $\mathbb{M} \mapsto \mathbb{M}|\varphi$ can be found in van Ditmarsch et al (2007), which provides a comprehensive introduction to this logical system. For our current purposes, it suffices to note that PAL has various tautologies describing the subtle interaction between knowledge and public announcements:

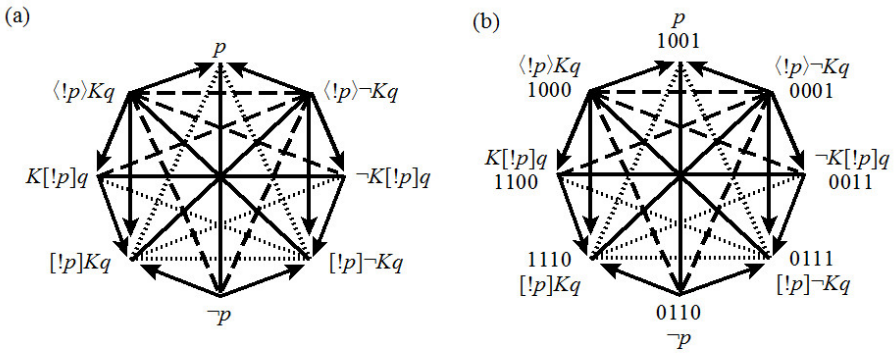

Fig. 10 Béziau octagon with (a) formulas from PAL, and (b) their bitstring representations.

Lemma 10 *It holds that* $\models_{\mathsf{PAL}} \langle !\varphi\rangle K\psi \rightarrow K[!\varphi]\psi$ *and* $\models_{\mathsf{PAL}} K[!\varphi]\psi \rightarrow [!\varphi]K\psi$.

The logical geometry of PAL has been studied in Demey (2012, 2014), where it is shown that this logical system gives rise to several interesting Aristotelian diagrams, such as the Béziau octagon[49] shown in Figure 10(a). A natural next move consists in defining a bitstring semantics for these diagrams. Letting $\mathscr{F}$ be the set of formulas occurring in the Béziau octagon in Figure 10(a), Demey (2012, 2014) has calculated the Boolean closure of $\mathscr{F}$, which turns out to contain 16 formulas. This strongly suggests that a bitstring semantics for $\mathscr{F}$ can indeed be given, by making use of bitstrings of length 4 (since $16 = 2^4$). However, because of the big conceptual differences between PAL and more 'classical' logical systems (such as CPL, FOL and S5), the original bitstring approach does not provide any insights as to how exactly these bitstrings should be associated with the formulas of $\mathscr{F}$. Finding a bitstring semantics for PAL-fragments such as $\mathscr{F}$ was thus left as an open problem in Demey (2012, 2014).

By contrast, using the bitstring approach presented in Section 3, we straightforwardly obtain a systematic bitstring semantics for the fragment $\mathscr{F}$, i.e. for the Béziau octagon in Figure 10(a). Applying Definition 5 and Lemma 10, we see that $\mathscr{F}$ induces the partition $\Pi_{\mathsf{PAL}}(\mathscr{F}) = \{\langle !p\rangle Kq, \neg p \wedge K[!p]q, \neg p \wedge \neg K[!p]q, \langle !p\rangle\neg Kq\}$. Note that $|\Pi_{\mathsf{PAL}}(\mathscr{F})| = 4$, so we will be working with bitstrings of length 4 (as expected). The bitstring mapping based on this partition is thus $\beta^{\mathscr{F}}_{\mathsf{PAL}} : \mathbb{B}(\mathscr{F}) \rightarrow \{0,1\}^4$. For example, for the formula $K[!p]q \in \mathscr{F}$ we have:

$$
\begin{array}{lrcrll}
\models_{\mathsf{PAL}} & \langle !p\rangle Kq & \rightarrow & K[!p]q & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{PAL}}(K[!p]q)]_1 = 1 \\
\models_{\mathsf{PAL}} & (\neg p \wedge K[!p]q) & \rightarrow & K[!p]q & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{PAL}}(K[!p]q)]_2 = 1 \\
\models_{\mathsf{PAL}} & (\neg p \wedge \neg K[!p]q) & \rightarrow & \neg K[!p]q & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{PAL}}(K[!p]q)]_3 = 0 \\
\models_{\mathsf{PAL}} & \langle !p\rangle\neg Kq & \rightarrow & \neg K[!p]q & \text{and thus} & [\beta^{\mathscr{F}}_{\mathsf{PAL}}(K[!p]q)]_4 = 0
\end{array}
$$

In sum, we find that $\beta^{\mathscr{F}}_{\mathsf{PAL}}(K[!p]q) = 1100$. Completely analogously, we can assign a bitstring $\beta^{\mathscr{F}}_{\mathsf{PAL}}(\varphi)$ to every formula $\varphi \in \mathscr{F}$, as shown in Figure 10(b).

[49] The Béziau octagon is an Aristotelian diagram named after Béziau (2003), and can be shown to contain a JSB hexagon and an SC hexagon.

The bitstring technique developed in Section 3 thus allows us to systematically assign bitstrings to the PAL-fragment $\mathscr{F}$. Furthermore, even though some work is needed to calculate the partition $\Pi_{\mathsf{PAL}}(\mathscr{F})$ induced by $\mathscr{F}$, it is *not* necessary to calculate the entire Boolean closure $\mathbb{B}(\mathscr{F})$.[50] Finally, the bitstring mapping $\beta^{\mathscr{F}}_{\mathsf{PAL}}$ assigns bitstrings to all formulas in $\mathbb{B}(\mathscr{F})$, including those not belonging to $\mathscr{F}$ itself. Consider, for example, the formula $\lambda := [!p]Kq \wedge (p \vee \neg K[!p]q)$, which is clearly a Boolean combination of $\mathscr{F}$-formulas, and cannot be rewritten into a syntactically simpler form. By noting that $\beta^{\mathscr{F}}_{\mathsf{PAL}}(\lambda) = 1010$, we can quickly determine the Aristotelian relations holding between λ and the formulas in $\mathscr{F}$; for example, λ is contrary to $\langle !p\rangle\neg Kq$, subcontrary to $[!p]\neg Kq$, and stands in no Aristotelian relation at all to $K[!p]q$ and p.

Finally, it should be noted that the formulas $\langle !p\rangle Kq$ and $[!p]Kq$, together with their negations, constitute a classical Aristotelian square embedded inside the Béziau octagon in Figure 10. However, this square looks quite strange, since the subalternation runs from the 'existential' formula $\langle !p\rangle Kq$ to the 'universal' formula $[!p]Kq$, rather than the other way around;[51] using Demey's (2012; 2014) terminology, it is a 'reversed square'. In Demey (2016) it is shown that similar reversed squares also arise in very different logical contexts, such as Russell's theory of definite descriptions and first-order logic interpreted over domains of at most one object. Furthermore, it is argued that these (families of) reversed squares ultimately point to a fundamental similarity between both areas, viz. the underlying existence of a partial function (in the case of PAL, this is the model update operation $\mathbb{M} \mapsto \mathbb{M}|\varphi$).[52]

This example (and the two others mentioned in Footnote 52) illustrates the powerful heuristic role that Aristotelian diagrams, and thus indirectly also bitstrings, can play in logical research: by compactly representing certain logical formulas and the Aristotelian relations holding between them, they allow us to explore unexpected connections between *prima facie* unrelated areas of logic. This provides some evidence for the claim that Aristotelian diagrams can function as a 'language' for the field of logic (and related fields such as linguistics and cognitive science),[53] in a manner that

[50] To better appreciate the difference between these two calculations, note that it follows from Theorems 4 and 5 that $|\Pi_{\mathsf{S}}(\mathscr{F})| \leq 2^{|\mathscr{F}|}$, whereas $|\mathbb{B}(\mathscr{F})| = 2^{|\Pi_{\mathsf{S}}(\mathscr{F})|} \leq 2^{(2^{|\mathscr{F}|})}$ (for any logical system S and fragment $\mathscr{F}$).

[51] Note the notational similarity between the public announcement operators $\langle !p\rangle/[!p]$ and the ordinary modal operators $\Diamond/\Box$, respectively. See Demey (2012, 2014) for a more precise explanation as to why $\langle !p\rangle$ is existential in nature and $[!p]$ universal.

[52] A similar situation arises in applications of Aristotelian diagrams in artificial intelligence. In a recent series of papers, Dubois, Prade and various co-authors have discovered that a single 'cube of opposition' (i.e. an Aristotelian octagon) can be used to describe various knowledge representation formalisms, such as formal concept analysis, modal logic, rough set theory, Sugeno integrals and several others. Consequently, they state that "This discovery leads to a new perspective on many knowledge representation formalisms, laying bare their underlying common features. The cube of opposition exhibits fruitful parallelisms between different formalisms, which leads to highlight some missing components present in one formalism and currently absent from another." (Dubois et al, 2015, p. 2933). Yet another, more subtle example can be found in Smessaert and Demey (2015b), which compares two competing accounts of the subjective quantifiers *many* and *few*. It is shown that on both accounts, these quantifiers induce a quadripartition (and thus require bitstrings of length 4) and yield a Buridan octagon, but one of them yields a better correlation between logical and lexical complexity, as reflected in the slightly different ways in which the subjective quantifier expressions are mapped onto the bitstrings.

[53] Recall the *lingua franca* quotation given in Section 1.

is roughly similar to the way in which category theory functions as a language for the field of mathematics (and related fields such as theoretical computer science and physics) (Landry, 1999; Pierce, 1991; Coecke and Paquette, 2011). Needless to say, developing these tentative remarks and analogies into a substantial account of the methodological role of Aristotelian diagrams in logic will require much more work than can be done here, but our goal in these last two paragraphs has merely been to sketch the possible outlines of such an account.

7 Concluding Thoughts

In this paper we have presented a technique for obtaining combinatorial bitstring semantics for arbitrary logical fragments. This technique involves defining the partition induced by a given fragment (Subsection 3.1), and then constructing the bitstring isomorphism based on that partition (Subsection 3.2). Furthermore, we have proved a number of theorems on the correlation between fragment size and minimal bitstring length (Subsection 3.3).

Although bitstrings have been used informally in recent years, yielding a wide variety of logical and diagrammatic results (Subsection 2.2), we have argued that this more informal approach suffers from a certain kind of arbitrariness, which is manifested in a number of problems, seriously restricting its applicability (Subsection 2.3). We have shown, however, that these problems are systematically overcome by the more formal bitstring approach developed in the present paper.[54]

The first problem concerns the sensitivity of the Aristotelian relations with respect to the specific properties of the underlying logical system. The new bitstring approach allows us to systematically assign logic-dependent bitstrings to formulas, and thus to deal with issues such as existential import in classical syllogistics versus contemporary first-order logic (Section 4). The second problem concerns the interplay between Boolean and Aristotelian structure: the latter is determined by the former, but not vice versa. By means of the new bitstring approach, we can divide an Aristotelian family into various subtypes that are Aristotelian but not Boolean isomorphic to each other (and thus have distinct Boolean closures), and study them in terms of the length of their minimal bitstring representations. Typical examples include the distinction between various subtypes of JSB hexagons (Subsection 5.1) and between various subtypes of Buridan octagons (Subsection 5.2). The third problem of the more informal bitstring approach concerns its apparent lack of systematicity. The new bitstring approach, however, provides a systematic strategy for assigning bitstrings to fragments of arbitrary logical systems, such as public announcement logic (Section 6).

[54] It should be emphasized that despite its fully formal nature, the bitstring approach developed here is also perfectly applicable to natural language sentences and expressions. (To appreciate the importance of this observation, recall from Section 1 that linguistics is one of the primary fields of application for Aristotelian diagrams, and indirectly thus also for bitstrings.) After all, the crucial idea of the present approach is that of a fragment inducing a *partition*, which is independent of whether that fragment consists of formal or natural language sentences (also see Footnote 22). For example, Seuren and Jaspers (2014) have shown that several elementary lexical fields (i.e. lexically coherent fragments of natural language expressions, such as {married, husband, wife, single}) induce tripartitions, and thus correspond to bitstrings of length 3. More recently, Roelandt (2016) has shown that other, more complex lexical fields (e.g. measure adjectives and gradable adjectives) induce quadripartitions, and thus correspond to bitstrings of length 4.

In ongoing work, we are studying the mathematical details of further generalizations of the bitstring approach, with a focus on cases that go beyond the Boolean realm. For example, even if the underlying logical system S has all the usual Boolean connectives, one might still wonder if, and to what extent, the bitstring mapping $\beta_{\mathsf{S}}^{\mathscr{F}}$ can be applied to formulas *outside* the Boolean closure of $\mathscr{F}$ (cf. Footnote 23). Demey (2017) investigates this question in detail, exploring connections with rough set theory (Yao, 2013). More radically, one might wonder whether the bitstring approach still works if the system S is no longer assumed to be Boolean in nature. For example, Ciucci et al (2016) have generalized the Aristotelian relations to several systems of many-valued logic, and it is currently an open question how to define an accompanying generalized bitstring semantics.[55]

In another direction, we are also applying the bitstring approach developed here to new fragments and logical systems (which are often of considerable historical or interdisciplinary interest), and exploring the phenomena that they give rise to. For example, using bitstrings we can straightforwardly determine the Boolean closure of the recently proposed Aristotelian cube for various knowledge representation formalisms (cf. Footnote 52), which previously had to be left as an open problem (Ciucci et al, 2016, Subsection 3.4). Furthermore, the interplay between Boolean and Aristotelian structure turns out to play a critical role in the study of Aristotelian diagrams proposed by the medieval Arabic philosopher Avicenna (Smessaert and Demey, 2015a), while the logic-sensitivity of Aristotelian diagrams is clearly illustrated by certain diagrams proposed by John N. Keynes and Hans Reichenbach (Demey, 2015). Finally, we plan to investigate diagrams that exhibit both phenomena—i.e. Boolean/Aristotelian interplay and logic-sensitivity—simulteanously.

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[55] If S is fully Boolean in nature, but also has additional language elements (e.g. modal operators, quantifiers, etc.), it might be interesting to develop a notion of bitstring semantics that also takes those additional elements into account (rather than just focusing on the Boolean structure of S). However, a potential disadvantage of such a fine-grained bitstring mapping might be that it remains too close to the system S itself, and will thus no longer allow us to draw interesting comparisons *across* different logical systems and applications (cf. Footnote 52). Thanks to an anonymous referee for an interesting discussion on this issue.

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